

m -CLUSTER TILTED ALGEBRAS OF TYPE $\tilde{\mathbb{A}}$

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ABSTRACT. In this paper, we characterize all the finite dimensional algebras that are m -cluster tilted algebras of type $\tilde{\mathbb{A}}$. We show that these algebras are gentle and we give an explicit description of their quivers with relations.

INTRODUCTION

Cluster categories were introduced in [9], and simultaneously in [14] for the case \mathbb{A} , in order to model the combinatorics of the cluster algebras of Fomin and Zelevinski [18], using tilting theory. The clusters correspond to the tilting objects in the cluster category.

Roughly speaking, given an hereditary finite dimensional algebra H over an algebraically closed field, the cluster category \mathcal{C}_H is obtained from the derived category $\mathcal{D}^b(H)$ by identifying the shift functor $[1]$ with the Auslander - Reiten translation τ . By a result of Keller [19], the cluster category is triangulated, and the same holds for the category obtained from $\mathcal{D}^b(H)$ by identifying the composition $[m] := [1]^m$ with τ . The latter is called an m -cluster category, in which m -cluster tilting objects have been defined by Thomas, in [22], who in addition showed that they are in bijective correspondence with the m -clusters associated by Fomin and Reading to a finite root system in [16]. The endomorphism algebras of the m -cluster tilting objects are called m -cluster tilted algebras or, in case $m = 1$, cluster tilted algebras.

Caldero, Chapoton and Schiffler gave in [14] an interpretation of the cluster categories \mathcal{C}_H in case H is hereditary of Dynkin type \mathbb{A} in terms of triangulations of the disc with marked points on its boundary, using an approach also present in [17] in a much more general setting. This has been generalized by several authors. For instance, Baur and Marsh in [7] considered m -angulations of the disc, modeling the m -cluster categories. In [1], Assem *et al.* showed that cluster tilted algebras coming from triangulations of the disc or the annulus with marked points on their boundaries are gentle, and, in fact, that these are the only gentle cluster tilted algebras. The class of gentle algebras defined by Assem and Skowroński in [4] has been extensively studied in [2, 6, 8, 12, 20, 21], for instance, and is particularly well understood, at least from the representation theoretic point of view. This class includes, among others, iterated tilted and cluster tilted algebras of types \mathbb{A} and $\tilde{\mathbb{A}}$, and, as shown in [21], is closed under derived equivalence. In [20], Murphy gives a description of the m -cluster tilted algebras of type \mathbb{A} in terms of quivers and relations. To do so, he works with the geometric realization of the m -cluster category of type \mathbb{A} made by Baur and Marsh in [7]. In [13] we give a classification up to derived equivalence of the m -cluster tilted algebras of type \mathbb{A} , see [8] for the case $m = 1$.

In the present paper, we completely classify the m -cluster tilted algebras of type $\tilde{\mathbb{A}}$ in terms of quivers and relations, using the geometric model proposed by Torkildsen in [23].

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We now state the main result of this paper (for the definitions of the terms used, we refer the reader to section 7 below).

Theorem. *A connected algebra $A = kQ/I$ is a connected component of an m -cluster tilted algebra of type $\tilde{\mathbb{A}}$ if and only if (Q, I) is a gentle bound quiver satisfying the following conditions:*

- (a) *It can contain a non-saturated cycle $\tilde{\mathcal{C}}$ in such a way that A is an algebra with root $\tilde{\mathcal{C}}$.*
- (b) *If $\tilde{\mathcal{C}}$ is an oriented cycle, then it must have at least one internal relation.*
- (c) *If the quiver contains more cycles, then all of them are m -saturated.*
- (d) *Outside of an m -saturated cycle it can contain at most $m - 1$ consecutive relations.*
- (e) *If there are internal relations in the root cycle, then the number of clockwise oriented relations is equal modulo m to the number of counterclockwise oriented.*

It is important to observe that m -cluster tilted algebras with $m \geq 2$ need not be connected and, as we shall see, their type is not uniquely determined, as we show below. This fact is essential in [15].

The paper is organized as follows: In section 1 we recall facts about gentle algebras, m -cluster tilted algebras and a geometric model of the m -cluster category of type $\tilde{\mathbb{A}}$. We use this section to fix some notation. In sections 2 and 3 we establish some properties of $(m + 2)$ -angulations and m -cluster categories that will be used in the sequel. In section 4 we study the m -cluster tilting objects. Sections 5 and 6 are devoted to the study of the bound quiver of an m -cluster tilted algebra of type $\tilde{\mathbb{A}}$. We continue in section 7 with some consequences, among which, we prove that m -cluster tilted algebras of type $\tilde{\mathbb{A}}$ are gentle.

1. PRELIMINARIES

1.1. Gentle algebras. While we briefly recall some concepts concerning bound quivers and algebras, we refer the reader to [3] or [5], for instance, for unexplained notions.

Let k be a commutative field. A *quiver* Q is the data of two sets, Q_0 (the *vertices*) and Q_1 (the *arrows*) and two maps $s, t: Q_1 \rightarrow Q_0$ that assign to each arrow α its *source* $s(\alpha)$ and its *target* $t(\alpha)$. We write $\alpha: s(\alpha) \rightarrow t(\alpha)$. If $\beta \in Q_1$ is such that $t(\alpha) = s(\beta)$ then the composition of α and β is the path $\alpha\beta$. This extends naturally to paths of arbitrary positive length. The *path algebra* kQ is the k -algebra whose basis is the set of all paths in Q , including one stationary path e_x at each vertex $x \in Q_0$, endowed with the multiplication induced from the composition of paths. In case $|Q_0|$ is finite, the sum of the stationary paths - one for each vertex - is the identity.

If the quiver Q has no oriented cycles, it is called *acyclic*. A *relation* in Q is a k -linear combination of paths of length at least 2 sharing source and target. A relation which is a path is called *monomial*, and the relation is *quadratic* if the paths appearing in it have all length 2. Let \mathcal{R} be a set of relations. Let $\langle Q_1 \rangle$ denote the two-sided ideal of kQ generated by the arrows, and I be the one generated by \mathcal{R} . Then $I \subseteq \langle Q_1 \rangle^2$. The ideal I is called *admissible* if there exists a natural number $r \geq 2$ such that $\langle Q_1 \rangle^r \subseteq I$. The pair (Q, I) is a *bound quiver*, and associated to it is the algebra $A = kQ/I$. It is known that any finite dimensional basic and connected algebra over an algebraically closed field is obtained in this way, see [3], for instance.

Recall from [4] that an algebra $A = kQ/I$ is said to be *gentle* if $I = \langle \mathcal{R} \rangle$, with \mathcal{R} a set of monomial quadratic relations such that :

- G1. For every vertex $x \in Q_0$ the sets $s^{-1}(x)$ and $t^{-1}(x)$ have cardinality at most two;
- G2. For every arrow $\alpha \in Q_1$ there exists at most one arrow β and one arrow γ in Q_1 such that $\alpha\beta \notin I$, $\gamma\alpha \notin I$;

G3. For every arrow $\alpha \in Q_1$ there exists at most one arrow β and one arrow γ in Q_1 such that $\alpha\beta \in I$, $\gamma\alpha \in I$.

Gentle algebras are special biserial (see [24]), and have been extensively studied in several contexts, see for instance [6, 8, 12, 20, 21].

1.2. m -cluster tilted algebras. Let $H \simeq \mathbf{k}Q$ be an hereditary algebra such that Q is acyclic. The derived category $\mathcal{D}^b(H)$ is triangulated, the translation functor, denoted by $[1]$, being induced from the shift of complexes. For an integer n , we denote by $[n]$ the composition of $[1]$ with itself n times, thus $[1]^n = [n]$. In addition, $\mathcal{D}^b(A)$ has Auslander-Reiten triangles, and, as usual, the Auslander-Reiten translation is denoted by τ .

Let m be a natural number. The m -cluster category of H is the quotient category $\mathcal{C}_m(H) := \mathcal{D}^b(H)/\tau^{-1}[m]$ which carries a natural triangulated structure, see [19]. In [9] the authors showed that the m -cluster category $\mathcal{C}_m(H)$ is *Calabi-Yau of CY-dimension $m+1$* (shortly $(m+1)$ -CY), that is, there is a bifunctorial isomorphism $\text{Hom}_{\mathcal{C}_m(H)}(X, \tau Y[1]) \cong D\text{Hom}_{\mathcal{C}_m(H)}(Y, X)$.

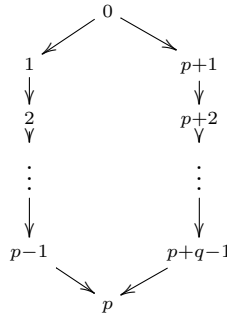
Following [22] we consider *m -cluster tilting objects* in $\mathcal{C}_m(H)$ defined as objects satisfying the following conditions:

- (1) $\text{Hom}_{\mathcal{C}_m(H)}(T, X[i]) = 0$ for all $i \in \{1, 2, \dots, m\}$ if and only if $X \in \text{add } T$,
- (2) $\text{Hom}_{\mathcal{C}_m(H)}(X, T[i]) = 0$ for all $i \in \{1, 2, \dots, m\}$ if and only if $X \in \text{add } T$.

The endomorphism algebras of such objects are called *m -cluster tilted algebras of type Q* . In case $m = 1$, this definition specializes to that of a cluster tilted algebra, a class intensively studied since its definition in [10].

In [1] it has been shown that cluster tilted algebras are gentle if and only if they are of type \mathbb{A} or $\tilde{\mathbb{A}}$. Using arguments similar to those of [1], Murphy showed in [20] that m -cluster tilted algebras, with $m \geq 2$, are also gentle and that the only cycles that may exist are cycles of length $(m+2)$, briefly called $(m+2)$ -cycles. Each of these cycles has full relations, that is, the composition of any two arrows on the cycle is zero. Perhaps the most noticeable differences between cluster tilted and m -cluster tilted algebras is that the latter need not to be connected, that the quiver does not determine the algebras and that there is no uniqueness of type like we will show in section 7.

Throughout the rest of this paper, \mathcal{C}_m denotes the m -cluster category $\mathcal{C}_m(H)$ with H an hereditary algebra of type $\tilde{\mathbb{A}}$ and we fix the following numbering and orientation for the quiver $\tilde{\mathbb{A}}$ (or $\tilde{\mathbb{A}}_{p,q}$):



where $p, q \geq 1$.

Recall that the Auslander-Reiten quiver of \mathcal{C}_m contains m transjective components \mathcal{S}^d , m tubular components \mathcal{T}_p^d of rank p , m tubular components \mathcal{T}_q^d of rank q , where $d \in \{0, \dots, m-1\}$, and infinitely many tubular components of rank 1.

1.3. A geometric realization of the m -cluster category of type $\tilde{\mathbb{A}}$. We follow [23]. Let $m \geq 1$ and $p, q \geq 2$ be integers. Let $P_{p,q,m}$ be a regular mp -gon, with a regular mq -gon at its center, cutting a hole in the interior of the outer polygon. Denote by $P_{p,q,m}^0$ the interior between the outer and inner polygon. Label the vertices on the outer polygon $O_0, O_1, \dots, O_{mp-1}$ in the counterclockwise direction, and label the vertices of the inner polygon $I_0, I_1, \dots, I_{mq-1}$, in the clockwise direction.

Let $\delta_{i,k}$ (or $\gamma_{i,k}$) be the path in the counterclockwise (or clockwise, respectively) direction from O_i (or I_i) to O_{i+k-1} (or I_{i+k-1} , respectively) along the border of the outer (or inner) polygon, where k is the number of vertices that the path runs through (including the start and end vertex).

We define two paths to be *equivalent* if they start in the same vertex, end in the same vertex and they are homotopic. We call these equivalence classes diagonals in $P_{p,q,m}$. Let $O_{i,t}$ denote the diagonals homotopic to $\delta_{i,t}$, and let $I_{i,t}$ be the diagonals homotopic to $\gamma_{i,t}$.

An m -diagonal in $P_{p,q,m}$ of type 1 is a diagonal between O_i and I_j with i congruent to j modulo m . An m -diagonal of type 2 (or type 3) is a diagonal of the form $O_{i,km+2}$ (or $I_{i,km+2}$, respectively), with $k \geq 1$ and $i \in \{0, \dots, pm-1\}$ (or $i \in \{0, \dots, qm-1\}$, respectively).

We say that a set of m -diagonals *cross* if they intersect in the interior $P_{p,q,m}^0$. A set of non-crossing m -diagonals that divides $P_{p,q,m}$ into $(m+2)$ -gons is called an $(m+2)$ -angulation. Torkildsen [23, Prop. 4.2] shows that the number of m -diagonals in any $(m+2)$ -angulation of $P_{p,q,m}$ is exactly $p+q$.

1.3.1. The quiver corresponding to an $(m+2)$ -angulation. For any $(m+2)$ -angulation Δ of $P_{p,q,m}$, Torkildsen [23] defines a corresponding coloured quiver Q_Δ with $p+q$ vertices in the following way. The vertices are the m -diagonals. There is an arrow between i and j if the m -diagonals bound a common $(m+2)$ -gon. The colour of the arrow is the number of edges forming the segment of the boundary of the $(m+2)$ -gon which lies between i and j , counterclockwise from i . This is the same definition as in [11] in the Dynkin \mathbb{A} case, and it is easy to see that such a quiver satisfies the conditions described there for coloured quivers.

Let Q_Δ^0 be the subquiver of the coloured quiver associated to the $(m+2)$ -angulation Δ given by the arrows of colour 0. The idea of the present article is to show that all m -cluster tilted algebras of type $\tilde{\mathbb{A}}$ are given by an $(m+2)$ -angulation Δ of the polygon $P_{p,q,m}$ in such a way that the ordinary quiver of the algebra is the quiver Q_Δ^0 and the relations are also determined by the $(m+2)$ -angulation. This description allows us to show easily that m -cluster tilted algebras of type $\tilde{\mathbb{A}}$ are gentle.

1.3.2. The category of m -diagonals. Let α be an m -diagonal in $P_{p,q,m}$ and s a positive integer. We define $\alpha[s]$ to be the m -diagonal obtained by rotating the outer polygon s steps clockwise and the inner polygon s steps counterclockwise. Also we have the opposite operation that we denote by $[-s]$. We set $\tau = [m]$.

We define the *category of m -diagonals* $\mathcal{C}_{p,q}^m$ as the \mathbf{k} -linear additive category having as indecomposable objects the m -diagonals, so the objects are the finite sets of m -diagonals. The space of morphisms between the indecomposable objects X and Y is the vector space over \mathbf{k} spanned by the elementary moves (morphisms defined in such a way that they correspond to irreducible morphisms in \mathcal{C}_m) modulo certain mesh relations. See [23, Section 6] for more details.

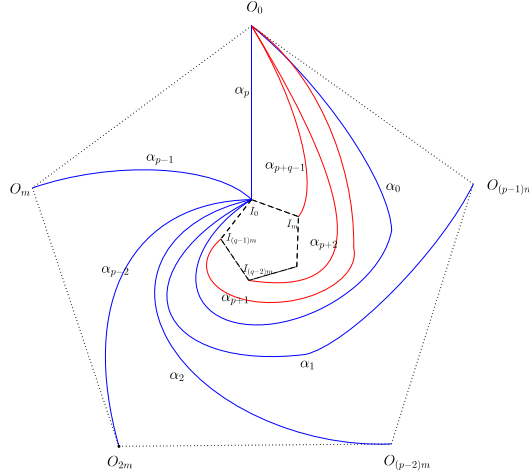
Let α be an m -diagonal. If α is of type 1, we say that α has *level* d (with $0 \leq d < m$) if we can reach any m -diagonal between O_d and I_d from α by applying a finite sequence of τ and elementary moves. If α is of type 2 (or 3), we say that α has level d if we can reach the m -diagonal $O_{d+1,m+2}$ (or $I_{d+1,m+2}$, respectively) from α by applying a finite sequence of τ and elementary moves.

Given the category of m -diagonals $\mathcal{C}_{p,q}^m$ we define the *Auslander-Reiten quiver* of $\mathcal{C}_{p,q}^m$ in the following way. The vertices are the indecomposable objects (that is, the m -diagonals), and there is an arrow from the indecomposable object α to the indecomposable object β if and only if there is an elementary move $\alpha \rightarrow \beta$.

Following the same notation as in [23], we denote by \mathcal{T}_p^d (or \mathcal{T}_q^d), the tube of rank p (or q , respectively) of degree d in the m -cluster category \mathcal{C}_m and by \mathcal{S}^d the component in the Auslander-Reiten quiver of \mathcal{C}_m consisting of objects of the form $\tau^s P[d]$ where P is projective. Similarly let T_p^d (or T_q^d) be the component in the Auslander-Reiten quiver of $\mathcal{C}_{p,q}^m$ containing the objects of type 2 (or 3, respectively) in level d and S^d the component consisting of m -diagonals of type 1 in level d .

Now, we want to define an additive functor $F : \mathcal{C}_{p,q}^m \rightarrow \mathcal{C}_m$ in such a way that F induces a quiver isomorphism between the Auslander-Reiten quiver of $\mathcal{C}_{p,q}^m$ and a subquiver of the Auslander-Reiten quiver of \mathcal{C}_m .

We start defining a particular $(m+2)$ -angulation Δ_P of $P_{p,q,m}$. This $(m+2)$ -angulation is such that the image $F(\Delta_P) = \bigoplus_{i=0}^{p+q-1} P_i$, where P_i is the projective corresponding to the vertex i in the quiver. Thus, $\Delta_P = \{\alpha_0, \dots, \alpha_{p+q-1}\}$ as in the figure below.



Defining $F(\tau^t \alpha_i[d]) = \tau^t P_i[d]$ we get a bijection between the set of m -diagonals of type 1 and the set of indecomposable objects in the transjective components in the Auslander-Reiten quiver of \mathcal{C}_m .

We denote the p indecomposable objects on the mouth (the quasi-simples) of the tube \mathcal{T}_p^d of rank p by $M_i[d]$ (for $i \in \{0, \dots, p-1\}$) where

$$M_0 := \begin{array}{c} 0 \\ p+1 \\ \vdots \\ p \end{array} \quad \text{and} \quad M_k := S_{p-k} \quad \text{if } 1 \leq k < p.$$

The other indecomposable objects that are in levels greater than 1 of the tube are denoted by $Q_i^s[d]$ where s is the quasi-length. For a given i , a *ray* is an infinite sequence of irreducible maps

$$M_i[d] \rightarrow Q_i^2[d] \rightarrow Q_i^3[d] \rightarrow Q_i^4[d] \rightarrow \dots,$$

From now on, we also fix a notation for the diagonals of type 2 (or 3) depending on the degree d of the tube T_p^d (or T_q^d , respectively) and the level k that the diagonal has in the corresponding tube. Then, the p diagonals that are in the level k of the tube T_p^d are parametrized by $O_{mx-(d+1), km+2}$ with $x \in \{1, \dots, p\}$ and $k \geq 1$. Thus, we define $F(O_{mx-(d+1), km+2}) = M_{x+1}[d]$ and for $k \geq 2$, $F(O_{mx-(d+1), km+2}) = Q_{x+1}^k[d]$ (where we compute the sub-index always modulo p). Therefore we get a bijection between the set of m -diagonals of type 2 and the set of indecomposable objects in the tubular component \mathcal{T}_p^d in the Auslander-Reiten quiver of \mathcal{C}_m .

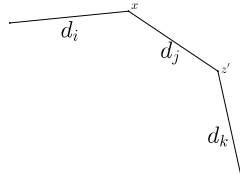
We do similarly with the m -diagonals of type 3 and they correspond to objects in the tubes of rank q . Checking that elementary moves (arrows in the quiver of $\mathcal{C}_{p,q}^m$) correspond to irreducible morphisms in \mathcal{C}_m we have:

1.1. Proposition. [23]

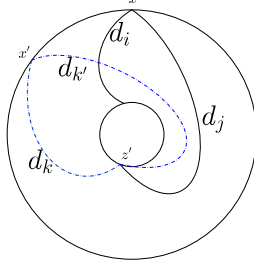
- (1) The component S^d in the Auslander-Reiten quiver of $\mathcal{C}_{p,q}^m$ is isomorphic (via F) to the component S^d in the Auslander-Reiten quiver of the m -cluster category \mathcal{C}_m of type $\tilde{\mathbb{A}}$, for all integers d , with $0 \leq d < m$.
- (2) The components T_p^d and T_q^e of the quiver of $\mathcal{C}_{p,q}^m$ are isomorphic (via F) to \mathcal{T}_p^d and \mathcal{T}_q^e respectively in the m -cluster category.

2. SOME RESULTS ABOUT $(m+2)$ -ANGULATIONS

2.1. Lemma. Let d_i , d_j and d_k be m -diagonals which are part of the same $(m+2)$ -gon in the $(m+2)$ -angulation Δ . Assume that they are arranged in such a way that d_i and d_j share a vertex of $P_{p,q,m}$, d_j and d_k share another vertex of $P_{p,q,m}$ but d_i , d_j and d_k do not have vertices in common (See figure below). Then d_i , d_j and d_k cannot be simultaneously m -diagonals of type 1.



Proof. We assume $d_i \cap d_j = \{x\}$ with x a vertex of the outer mp -polygon. Since the three diagonals delimit the same $(m+2)$ -gon, the diagonals d_j and d_k share a vertex that must not be x . Then, $d_j \cap d_k = \{z' = t(d_j)\}$ with z' in the inner mq -polygon. Therefore $t(d_k) = z'$ and $s(d_k) = x'$ with $x' \neq x$. We have two possibilities for d_k depending on the direction it takes.



If d_k is a diagonal between x' and z' in the clockwise sense, it is clear that $d_k \cap d_i \neq \emptyset$, which is impossible. If instead d_k is a diagonal between x' and z' in the counterclockwise sense we will have an arrow $d_k \rightarrow d_j$ instead of an arrow $d_j \rightarrow d_k$ in Q_Δ^0 . In consequence, it is impossible for the three diagonals to be of type 1. \square

2.2. Lemma. *Let d and d' be m -diagonals of type 2 (or 3). Assume that d belongs to the component T_p^i (or T_q^i) of the category of m -diagonals $\mathcal{C}_{p,q}^m$. If the source $s(d')$ of d' is equal to the target $t(d)$ of d , then d' belongs to the component T_p^{i-1} (or T_q^{i-1} , respectively).*

Proof. Since $d \in T_p^i$, $d = O_{mx-(i+1), km+2}$ with $x \in \{1, \dots, p\}$ and $k \geq 1$. Then, $s(d) = mx - (i+1)$ and $t(d) = mx - (i+1) + km + 1 = m(x+k) - i$. If $s(d') = t(d)$ we have $d' = O_{m(x+k)-i, k'm+2}$ with $k' \geq 1$. Therefore $d' \in T_p^{i-1}$ as claimed. \square

An easy consequence of this lemma is the following corollary.

2.3. Corollary. *Let \mathcal{C} be a component of the Auslander-Reiten quiver of the category $\mathcal{C}_{p,q}^m$ and let d and d' be two m -diagonals of type 2 (or type 3) belonging to \mathcal{C} . Then the target of the diagonal d cannot be the source of the diagonal d' reciprocally.*

3. THE CATEGORY \mathcal{C}_m

In this section we expose some easy results about morphisms in \mathcal{C}_m and we fix some notation.

3.1. Lemma. *Let H be an hereditary algebra and $m > 2$. Then $\text{Hom}_{\mathcal{C}_m(H)}(M, N[2]) = 0$ for all $M, N \in \text{mod} H$.*

Proof. By definition,

$$\text{Hom}_{\mathcal{C}_m(H)}(M, N[2]) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(H)}(M, F^i N[2]) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(H)}(M, \tau^{-i} N[im + 2]).$$

All the terms $\text{Hom}_{D^b(H)}(M, \tau^{-i} N[im + 2])$ with $im + 2 < 0$ are zero because the objects M and $\tau^{-i} N$ belong to $(\text{mod} H)[0]$. In addition, all the terms $\text{Hom}_{D^b(H)}(M, \tau^{-i} N[im + 2])$ with $im + 2 \geq 2$

are zero for H hereditary. To conclude, since $m > 2$, we cannot have $im + 2 = 0$ or $im + 2 = 1$ and hence $\text{Hom}_{\mathcal{C}_m(H)}(M, N[2]) = 0$. \square

3.2. Lemma. *Let X be an object in the tubular component \mathcal{T}_p^x and Y an object in the tubular component \mathcal{T}_q^x , with $p \neq q$. Then $\text{Hom}_{\mathcal{C}_m}(X, Y[1]) = 0$.*

Proof. Using the $m + 1$ -Calabi-Yau property of \mathcal{C}_m we get

$$\text{Hom}_{\mathcal{C}_m}(X, Y[1]) \cong D \text{Hom}_{\mathcal{C}_m}(\tau^{-1}Y, X) \cong D \text{Hom}_{\mathcal{C}_m}(Y, \tau X) = 0$$

which is zero because there are no morphisms between elements in different tubes. \square

4. THE m -CLUSTER TILTING OBJECTS IN \mathcal{C}_m

4.1. Definition. Let \mathcal{C} be a component of the Auslander-Reiten quiver of \mathcal{C}_m and M an object in \mathcal{C} . Then we define the following subsets:

$$M^r = \{N \in \mathcal{C} \mid \text{Hom}_{\mathcal{C}_m}(M, N) \neq 0\}$$

$$M^l = \{N \in \mathcal{C} \mid \text{Hom}_{\mathcal{C}_m}(N, M) \neq 0\}$$

Given an object M in \mathcal{C} the set $M^r \setminus (\tau^{-1}M)^r$ determine two rays starting in M . By convention we call M^{ru} the ray which is located above and M^{dr} the ray which is below.

In the same way, the set $M^l \setminus (\tau M)^l$ determine two rays ending in M . By convention we call M^{lu} the ray which is located above and M^{dl} the ray which is below.

4.2. Convention. Let P_p be the projective associated to the vertex p . We adopt the convention that P_p^{dr} is the ray

$$P_p \rightarrow P_{p-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0$$

and that P_p^{ur} is the ray

$$P_p \rightarrow P_{p+q-1} \rightarrow \cdots \rightarrow P_{p+1} \rightarrow P_0$$

4.3. Remark. Observe that in the component of \mathcal{C}_m isomorphic to S^0 the parallel rays to $P_p \rightarrow P_{p-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0$ are in bijection (via F^{-1}) with the diagonals that share a vertex in the inner polygon.

In the same way, the parallel rays to $P_p \rightarrow P_{p+q-1} \rightarrow \cdots \rightarrow P_{p+1} \rightarrow P_0$ are in bijection (via F^{-1}) with the diagonals that share a vertex in the outer polygon.

In the following lemmas we are going to see the relationship between two indecomposables summands of an m -cluster tilting object T if both are in the same component.

4.4. Lemma. *Let T be an m -cluster tilting object and let M, N be two objects in a transjective component. Then,*

- (1) *If $M \in \text{add}(T)$, then $\tau^{-i}M \notin \text{add}(T)$ for all $i \in \mathbb{N}^*$.*
- (2) *If $M, N \in \text{add}(T)$ and there is a nonzero morphism $M \rightarrow N$, then $N \in M^{ur} \cup M^{dr}$.*

Proof. (1) In the m -cluster category $[m] = \tau$ and consequently, for all $i \in \mathbb{N}^*$ we have:

$$\mathrm{Hom}_{\mathcal{C}_m}(M, \tau^{-i}M[m]) \cong \mathrm{Hom}_{\mathcal{C}_m}(M, \tau^{-(i-1)}M) \neq 0.$$

Thus, $\tau^{-i}M \notin \mathrm{add}(T)$.

(2) Since there is a nonzero morphism $M \rightarrow N$, we know that $N \in M^r$. Suppose that $N \notin M^{ur} \cup M^{dr}$. Then, $N = \tau^{-i}N'$ with $N' \in M^{ur} \cup M^{dr}$ or $N = \tau^{-i}M$, $i \in \mathbb{N}^*$. By 1, we cannot have $N = \tau^{-i}M$. Thus, $N = \tau^{-i}N'$ with $N' \in M^{ur} \cup M^{dr}$ and

$$\mathrm{Hom}_{\mathcal{C}_m}(M, \tau^{-i}N'[m]) \cong \mathrm{Hom}_{\mathcal{C}_m}(M, \tau^{-(i-1)}N') \neq 0.$$

In consequence $N = \tau^{-i}N' \notin \mathrm{add}(T)$, which contradicts our assumption. \square

4.5. Definition. Let \mathcal{C} be a tubular component of the Auslander-Reiten quiver of \mathcal{C}_m and \mathcal{N} a subset of \mathcal{C} . For any $r \in \mathbb{N}^*$, we define the subset $\mathcal{N}_{\leq r}$ by:

$$\mathcal{N}_{\leq r} = \{N \in \mathcal{N} \mid N \text{ belongs to the first } r \text{ levels of the tube}\}$$

4.6. Lemma. Let T be an m -cluster tilting object and let $M \in \mathrm{add}(T)$ be an object in the tubular component of rank p . Then, M belongs to the first $p-1$ levels of the tube.

Proof. Observe that if d_M is a diagonal of type 2 such that $F(d_M) = M$ then d_M has the form $O_{*, km+2}$, where k is the level of the diagonal in the tube T_p . If $k \geq p$, $km+2 \geq pm+2$. Then, the diagonal has a self-intersection and cannot be part of an $(m+2)$ -angulation. In consequence M cannot be an m -cluster tilting object summand. \square

4.7. Lemma. Let T be an m -cluster tilting object and let M, N be two objects in a tubular component of rank p . If $M, N \in \mathrm{add}(T)$ and there is a nonzero morphism $M \rightarrow N$, then $N \in (M^{ur} \cup M^{dr})_{\leq p-1}$.

Proof. As in the proof of lemma 4.4 we show that $N \in M^{ur} \cup M^{dr}$. It remains to prove that N belongs to the first $p-1$ levels of the tube, but this follows from the previous lemma. \square

Now, we want to study the relations between two summands of an m -cluster tilting object when both are in different components.

4.8. Lemma. Let T be an m -cluster tilting object. Assume that $P_a \in \mathrm{add}(T)$ and $\tau^r P_b[1] \in \mathrm{add}(T)$. Then $\mathrm{Hom}_{\mathcal{C}_m}(P_a, \tau^r P_b[1]) = 0$.

Proof. Since $P_a, \tau^r P_b[1] \in \mathrm{add}(T)$ and T is an m -cluster tilting object we have

$$0 = \mathrm{Hom}_{\mathcal{C}_m}(\tau^r P_b[1], P_a[1]) \cong \mathrm{Hom}_{\mathcal{C}_m}(P_b, \tau^{-r} P_a)$$

and then $\tau^{-r} P_a \notin P_b^r$. Otherwise, if $0 \neq \mathrm{Hom}_{\mathcal{C}_m}(P_a, \tau^r P_b[1])$ using the $(m+1)$ -CY property of \mathcal{C}_m we obtain $0 \neq D \mathrm{Hom}_{\mathcal{C}_m}(\tau^r P_b, \tau P_a) \cong D \mathrm{Hom}_{\mathcal{C}_m}(P_b, \tau^{1-r} P_a)$ and $\tau(\tau^{-r} P_a) \in P_b^r$, a contradiction. \square

To continue we need the following definitions.

4.9. Definition. Let \mathcal{C} be a tubular component of the Auslander-Reiten quiver of \mathcal{C}_m and let M be an object in \mathcal{C} . Then we define the following subsets:

$$\begin{aligned} M^+ &= \{N \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}_m}(M, N) \neq 0\}_{\leq p-1} \\ M^- &= \{N \in \mathcal{C} \mid \mathrm{Hom}_{\mathcal{C}_m}(N, M) \neq 0\}_{\leq p-1} \end{aligned}$$

In the same way we can define the same subsets if \mathcal{C} is a component of the category of m -diagonals $\mathcal{C}_{p,q}^m$.

4.10. Definition. Let M be an object belonging to the mouth of a tube of rank p in \mathcal{C}_m or $\mathcal{C}_{p,q}^m$. We define the *cone* \widehat{M} of M to be the set

$$\widehat{M} = \left(\bigcup_{N \in M^{dr}} N^{dl} \right)_{\leq p-1}$$

4.11. Lemma. Let S_i be the simple associated to the vertex i of the quiver and let M be an indecomposable object in the same tube as S_i . Then:

- (1) If $i \in \{1, \dots, p-1\}$, $\dim \text{Hom}_{\mathcal{C}_m}(P_i, M) = \begin{cases} 1, & \text{if } M \in \widehat{S}_i; \\ 0, & \text{otherwise.} \end{cases}$
- (2) If $i \in \{0, p, p+1, \dots, p+q-1\}$, $\dim \text{Hom}_{\mathcal{C}_m}(P_i, M) = \begin{cases} 1, & \text{if } M \in \widehat{M}_0; \\ 0, & \text{otherwise.} \end{cases}$

Proof. To prove this it is enough to see that the simple S_i appears just once as a composition factor of M if $M \in \widehat{S}_i$ and does not appear outside the cone \widehat{S}_i . For the quasi-simple module M_0 , the simples S_0, S_p and S_j (with $j \in \{p+1, \dots, p+q-1\}$) appear once in the cone \widehat{M}_0 and do not appear in the complement. The others simples do not appear. \square

4.12. Lemma. Let T be an m -cluster tilting object and $P_b, N \in \text{add}(T)$ such that N is an object in the tubular component \mathcal{T}_p . Suppose that $\text{Hom}_{\mathcal{C}_m}(P_b, N) \neq 0$, then $N \in M_*^{dl}$ where

$$* = \begin{cases} p-b, & \text{if } b \in \{1, \dots, p-1\}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since $\text{Hom}_{\mathcal{C}_m}(P_b, N) \neq 0$, by remark 4.11, we know that $N \in \widehat{M}_*$ with $*$ as above. In addition, since $P_b, N \in \text{add}(T)$, we have $0 = \text{Hom}_{\mathcal{C}_m}(N, P_b[1]) \cong D \text{Hom}_{\mathcal{C}_m}(P_b, \tau N)$. Then $\tau N \in \widehat{M}_*^c$ and $N \in (\tau^{-1} \widehat{M}_*)^c = (\widehat{M}_{*-1})^c$. In consequence $N \in \widehat{M}_* \cap (\widehat{M}_{*-1})^c = M_*^{dl}$. \square

4.13. Lemma. Let T be an m -cluster tilting object and $N, \tau^r P_b[1] \in \text{add}(T)$ such that N is an object in the tubular component \mathcal{T}_p . Assume that $\text{Hom}_{\mathcal{C}_m}(N, \tau^r P_b[1]) \neq 0$, then $N \in M_{*-1+r}^{dr}$ with $*$ as in the previous lemma (and we compute $*-1+r$ modulo p).

Proof. Since $0 \neq \text{Hom}_{\mathcal{C}_m}(N, \tau^r P_b[1]) \cong D \text{Hom}_{\mathcal{C}_m}(\tau^r P_b, \tau N) \cong D \text{Hom}_{\mathcal{C}_m}(P_b, \tau^{1-r} N)$ we have $\tau(\tau^{-r} N) \in \widehat{M}_*$. In the other hand, since $\tau^r P_b[1], N \in \text{add}(T)$ we have

$$0 = \text{Hom}_{\mathcal{C}_m}(\tau^r P_b[1], N[1]) \cong \text{Hom}_{\mathcal{C}_m}(P_b, \tau^{-r} N).$$

Then $\tau^{-r} N \in (\widehat{M}_*)^c$ and consequently $\tau^{-r} N \in (\widehat{M}_*)^c \cap \tau^{-1} \widehat{M}_* = M_{*-1}^{dr}$ and $N \in \tau^r M_{*-1}^{dr} = M_{*-1+r}^{dr}$. \square

In order to simplify the notation in the following lemma we introduce the next definition.

4.14. Definition. Let B be an object in a tube of rank p wich lies on the level $v \in \{1, \dots, p-1\}$ of the tube. The index $\alpha(B)$ of B is defined by $\alpha(B) = p-1-v$.

4.15. Lemma. *Let T be an m -cluster tilting object and $B, D \in \text{add}(T)$ such that $B \in \mathcal{T}_p^i$ and $D \in \mathcal{T}_p^{i+1}$. Then, $\text{Hom}_{\mathcal{C}_m}(B, D) \neq 0$ if and only if $D \in ((\mathcal{B}^-(\tau B)[1])^{dl})_{\leq \alpha(B)}$, and the level v of B is different from $p - 1$.*

Proof. Since $D \in \mathcal{T}_p^{i+1}$, there is $D' \in \mathcal{T}_p^i$ such that $D = D'[1]$. On the other hand, $B, D \in \text{add}(T)$ implies that

$$0 = \text{Hom}_{\mathcal{C}_m}(D'[1], B[1]) \cong \text{Hom}_{\mathcal{C}_m}(D', B)$$

and

$$0 = \text{Hom}_{\mathcal{C}_m}(D'[1], B[2]) \cong \text{Hom}_{\mathcal{C}_m}(D', B[1]) \cong D \text{Hom}_{\mathcal{C}_m}(B, \tau D').$$

Thus, $D' \in (B^- \cup (\tau^{-1}B)^+)^c$. If in addition we want that

$$\text{Hom}_{\mathcal{C}_m}(B, D) \cong \text{Hom}_{\mathcal{C}_m}(B, D'[1]) \cong D \text{Hom}_{\mathcal{C}_m}(D', \tau B)$$

be non-zero, we get $D' \in (\tau B)^- \setminus (B^- \cup (\tau^{-1}B)^+) = ((\mathcal{B}^-(\tau B)[1])^{dl})_{\leq \alpha(B)}$. \square

To each diagonal d in a tubular component of $\mathcal{C}_{p,q}^m$, we associate two diagonals (which may coincide) in the mouth of the tube according to the following definition.

4.16. Definition. Let d be a diagonal of type 2 or 3. We define $\mathcal{B}(d)$ (or $\mathcal{B}^-(d)$) as the unique diagonal that belongs to the intersection of d^{ur} (or d^{ul} , respectively) and the mouth of the tube.

In the same way we define $\mathcal{B}(T)$ and $\mathcal{B}^-(T)$ if T is an object in a tubular component of \mathcal{C}_m .

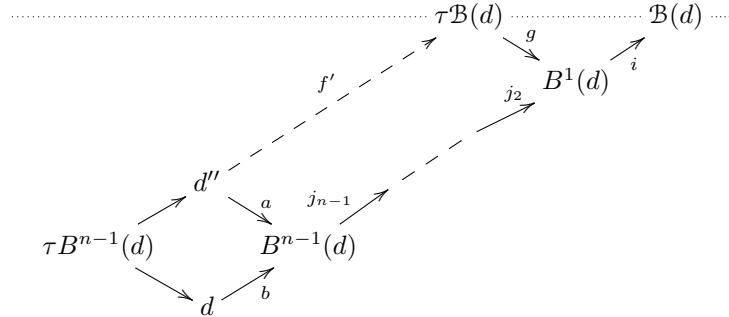
The following proposition (and its dual) show the importance of the objects that we have just defined.

4.17. Proposition. *Let d' be a diagonal of type 1 and let d be a diagonal of type 2 (or 3). If $\text{Hom}_{\mathcal{C}_m}(F(d'), F(d)) = 0$, then $\text{Hom}_{\mathcal{C}_m}(F(d'), F\mathcal{B}(d)) = 0$.*

Proof. It is enough to show that we can factorize the morphism $F(d') \rightarrow F(\mathcal{B}(d))$ through the morphism $F(d') \rightarrow F(d)$. We proceed inductively on the level k of the diagonal $d = O_{*,km+2}$ (or its image $F(d)$) on the tube T_p (or \mathcal{T}_p , respectively). If $k = 1$ the diagonal is in the mouth of the tube and thus $\mathcal{B}(d) = d$. Assume now the statement holds true for $k \leq n$ and let d be a diagonal on the level $n + 1$. We label the elements on the ray d^{ur}

$$d \xrightarrow{b} B^{n-1}(d) \xrightarrow{j_{n-1}} B^{n-2}(d) \xrightarrow{j_{n-2}} \dots \xrightarrow{j_2} B^1(d) \xrightarrow{i} \mathcal{B}(d)$$

(where the index s is the level on the tube less 1). We have the following situation in the tube:



Set $j = j_2 \circ \cdots \circ j_{n-1}$. By the induction hypothesis there is $c : d' \rightarrow B^{n-1}(d)$ such that the diagram

$$\begin{array}{ccc} d' & \xrightarrow{\quad} & \mathcal{B}(d) \\ \searrow \exists c & & \nearrow i \circ j \\ & B^{n-1}(d) & \end{array}$$

commutes.

In addition, since $0 \rightarrow \tau B^{n-1}(d) \rightarrow d'' \oplus d \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} B^{n-1}(d)$ is an almost split sequence, there is a morphism $\begin{pmatrix} h \\ h' \end{pmatrix} : d' \rightarrow d'' \oplus d$ such that $c = \begin{pmatrix} a & b \end{pmatrix} \circ \begin{pmatrix} h \\ h' \end{pmatrix}$. Moreover, if we apply the induction hypothesis to the diagonal d'' , we obtain $f : d' \rightarrow d''$ such that the diagram

$$\begin{array}{ccc} d' & \xrightarrow{\quad} & \tau \mathcal{B}(d) \\ \searrow \exists f & & \nearrow f' \\ & d'' & \end{array}$$

commutes. Then, we have the following commutative diagram:

$$\begin{array}{ccccccc} & & & \begin{pmatrix} h \\ h' \end{pmatrix} & d' & & \\ & & & \swarrow & \downarrow c & & \\ 0 & \longrightarrow & \tau B^{n-1}(d) & \longrightarrow & d'' \oplus d & \xrightarrow{\begin{pmatrix} a & b \end{pmatrix}} & B^{n-1}(d) \longrightarrow 0 \\ & & & \downarrow \begin{pmatrix} f' & 0 \\ 0 & Id \end{pmatrix} & & & \downarrow j \\ & & & \tau \mathcal{B}(d) \oplus d & \xrightarrow{\begin{pmatrix} g & j \circ b \end{pmatrix}} & B^1(d) & \\ & & & & & \downarrow i & \\ & & & & & & \mathcal{B}(d) \end{array}$$

and the factorization

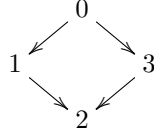
$$\begin{array}{ccc} d' & \xrightarrow{i \circ j \circ c} & \mathcal{B}(d) \\ \searrow \begin{pmatrix} h \\ h' \end{pmatrix} & & \nearrow \\ & d'' \oplus d & \end{array} \quad i \circ \begin{pmatrix} g & j \circ b \end{pmatrix} \circ \begin{pmatrix} f' & 0 \\ 0 & Id \end{pmatrix} = \begin{pmatrix} i \circ g \circ f' & i \circ j \circ b \end{pmatrix}$$

which is actually the factorization $d' \xrightarrow{\quad} \mathcal{B}(d)$ since $i \circ g = 0$.

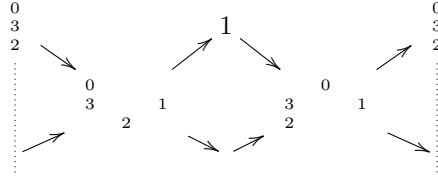
$$\begin{array}{ccc} d' & \xrightarrow{\quad} & \mathcal{B}(d) \\ \searrow h' & & \nearrow i \circ j \circ b \\ & d & \end{array}$$

□

4.18. Remark. Note that the converse is not true. Let A be the algebra of type $\tilde{\mathbb{A}}_{2,2}$ given by the quiver:



Then, one of the tubes of rank 2 of the Auslander-Reiten quiver of $\text{mod } A$ is the following



It is clear that $\text{Hom}_{\mathcal{C}_m}(2, 1) = 0$ but $\text{Hom}_{\mathcal{C}_m}(2, \begin{smallmatrix} 0 & & \\ 3 & & \\ & 1 & \\ 2 & & \end{smallmatrix}) \neq 0$.

5. THE ORDINARY QUIVER OF THE ALGEBRA $\text{End}_{\mathcal{C}_m}(F(\Delta))$

If $\Delta = \{d_1, \dots, d_{p+q}\}$ is an $(m+2)$ -angulation, then we denote by $F(\Delta) := F(d_1) \oplus \dots \oplus F(d_{p+q})$ the corresponding m -cluster tilting object.

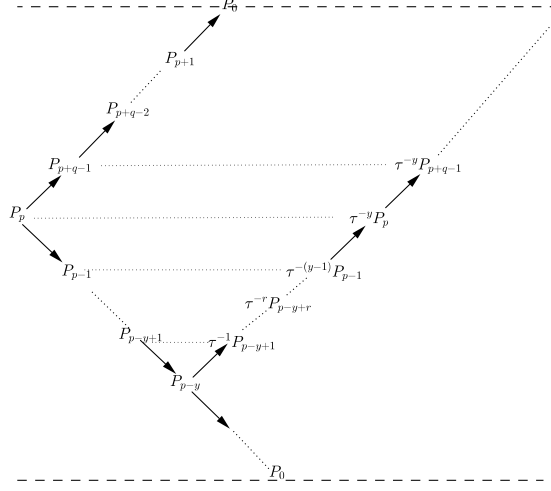
We write Q_Δ^0 the subquiver of the coloured quiver associated to the $(m+2)$ -angulation Δ given by the arrows of colour 0. Now, the idea is to compare the quiver Q_Δ^0 with the ordinary quiver of the endomorphism algebra $\text{End}_{\mathcal{C}_m}(F(\Delta))$.

5.1. Definition. Let x be a vertex in the outer polygon. We define $d_{x \curvearrowright}$ as the set of all the m -diagonals of type 1 starting at the vertex x .

5.2. Lemma. Let x be a vertex of the outer polygon (then $0 \leq x \leq mp - 1$) such that $x \equiv 0 \pmod{m}$. Then there exists $y \in \{1, \dots, p\}$ such that $x = m(p - y)$ and:

- (1) The diagonal α_{p-y} belongs to $d_{x \curvearrowright}$, and $d_{x \curvearrowright}$ belongs to the component S^0 .
- (2) $F(d_{x \curvearrowright}) = \{\tau^k P_{(p-y)-j} \mid k \equiv j \pmod{p}, j \in \{-y, \dots, p-y\}\} \cup \{\tau^{-y+kp} P_{p+j} \mid k \in \mathbb{Z}, j \in \{0, \dots, q\}\}$.

Proof. The existence of y is clear. Next, for 1 we simply observe that the diagonal α_{p-y} (if $y \in \{1, \dots, p\}$) is by definition a diagonal starting at $O_{m(p-y)=x}$ and finishing at I_0 and it belongs to the component S^0 . Then $F(d_{x \curvearrowright})$ is the following infinite ray containing the object $F(\alpha_{p-y}) = P_{p-y}$:

FIGURE 1. The ray $F(d_{x \curvearrowright})$

Consequently, $F(d_{x \curvearrowright}) = \{\tau^k P_{(p-y)-j} \mid k \equiv j \pmod{p}, j \in \{-y, \dots, p-y\}\} \cup \{\tau^{-y+kp} P_{p+j} \mid k \in \mathbb{Z}, j \in \{0, \dots, q\}\}$. \square

5.3. Remark. In general, if $x \equiv m-i \pmod{m}$ with $i \in \{0, \dots, m-1\}$ then there is $y \in \{1, \dots, p\}$ such that $x = m(p-y) - i$ and:

- (1) $d_{x \curvearrowright}$ belongs to the component S^i .
- (2) $F(d_{x \curvearrowright}) = \{\tau^k P_{(p-y)-j}[i] \mid k \equiv j \pmod{p}, j \in \{-y, \dots, p-y\}\} \cup \{\tau^{-y+kp} P_{p+j}[i] \mid k \in \mathbb{Z}, j \in \{0, \dots, q\}\}$.

\square

Now, we are ready to show the main result of this section.

5.4. Proposition. *Let Δ be an $(m+2)$ -angulation of $P_{p,q,m}$. The quiver $Q_{\text{End}_{\mathbb{C}_m}(F(\Delta))}$ associated to the algebra $\text{End}_{\mathbb{C}_m}(F(\Delta))$ is equal to the quiver Q_{Δ}^0 associated to the $(m+2)$ -angulation Δ .*

Proof. If d_i is a diagonal in the $(m+2)$ -angulation of $P_{p,q,m}$, we denote by T_i the indecomposable object of \mathbb{C}_m such that $F(d_i) = T_i$.

If two diagonals d_1 and d_2 share a vertex, there is a path between d_1 and d_2 in Q_{Δ}^0 . Assume that the path is oriented from d_1 to d_2 .

To begin, suppose that d_1 and d_2 are part of the same $(m+2)$ -gon and assume that both diagonals are of the same type 1 (or 2 or 3) sharing a vertex. Then d_1 and d_2 lie in the same component S^d (or T_p^d or T_q^d , respectively). Thus, since there is an arrow $d_1 \rightarrow d_2$ and the two diagonals do not intersect, we know that $d_2 \in d_1^{ur} \cup d_1^{dr}$. According to the isomorphism between the

components S^d and \mathcal{S}^d we have $T_2 \in T_1^{ur} \cup T_1^{dr}$. Then, it is clear that there is a morphism $T_1 \rightarrow T_2$ in \mathcal{C}_m .

Now, let us see the case where d_1 and d_2 are in two different $(m+2)$ -gons. Since both share a vertex of $P_{p,q,m}$ there must be a third diagonal d_3 having the same vertex in common. Hence, we have the arrows $d_1 \rightarrow d_3 \rightarrow d_2$ in Q_Δ^0 . Assume that d_3 is the same type as d_1 and d_2 . Since we have the arrows $d_1 \rightarrow d_3 \rightarrow d_2$ we know that $T_3 \in T_1^{ur} \cup T_1^{dr}$ and $T_2 \in T_3^{ur} \cup T_3^{dr}$. Suppose (without loss of generality) that $T_2 \in T_1^{ur}$. Then, if $T_3 \in T_1^{dr}$ we get that $T_2 \notin T_3^{ur} \cup T_3^{dr}$ which is impossible. In consequence, T_2 and T_3 belong to T_1^{ur} . Thus T_1, T_2 and T_3 lie in the same ray of the component \mathcal{S}^d and clearly there are morphisms $T_1 \rightarrow T_3 \rightarrow T_2$ in \mathcal{C}_m .

We continue with the case where d_1 is a diagonal of type 1 and d_2 is a diagonal of type 2 (or 3), both being part of the same $(m+2)$ -gon. In order to simplify the notation we assume that all diagonals are in degree 0. Then, we can write $d_1 = d_{xz}$ where $x = s(d_1)$, $z = t(d_1)$ and $x \equiv 0 \pmod{m}$ and $d_2 = O_{my-1, km+2}$ with $y \in \{1, \dots, p\}$ and $k \in \{1, \dots, p-1\}$. Since there is an arrow $d_1 \rightarrow d_2$, we know that d_2 follows d_1 and $t(d_2) = x$. Then, $t(d_2) = my - 1 + km + 1 \equiv x \pmod{mp}$ and so $m(k+y) \equiv x \pmod{mp}$. In addition, $x \equiv 0 \pmod{m}$ (since $d_1 \in S^0$), then $x = my'$ with $y' \in \{0, \dots, p-1\}$. Combining these we obtain $m(k+y-y') \equiv 0 \pmod{mp}$ and so $k+y \equiv y' \pmod{p}$. In consequence $\mathcal{B}(d_2) = O_{m(y+k-1)-1, m+2} = O_{m(y'-1)-1, m+2}$ and

$$F(\mathcal{B}(d_2)) = M_{y'} = \begin{cases} \begin{matrix} 0 \\ p+1 \\ \vdots \\ p \end{matrix}, & \text{if } y' \equiv 0 \pmod{p}; \\ S_{p-y'}, & \text{if } y' \not\equiv 0 \pmod{p}. \end{cases}$$

We want to prove that there is a nonzero morphism $T_1 \rightarrow T_2$ in \mathcal{C}_m . By proposition 4.17 it suffices to show that there is a nonzero morphism $T_1 \rightarrow M_{y'}$.

Since $T_1 \in F(d_{x\curvearrowright}) = \{\tau^k P_{p-y'-i} | k \equiv i \pmod{p}\} \cup \{\tau^{-y'+kp} P_{p+j} | k \in \mathbb{Z}, j \in \{0, \dots, q\}\}$, we have the following possibilities:

- If $T_1 \in \{\tau^k P_{p-y'-i} | k \equiv i \pmod{p}\}$:

$$\begin{aligned} \text{Hom}_{\mathcal{C}_m}(\tau^k P_{p-y'-i}, M_{y'}) &\cong \text{Hom}_{\mathcal{C}_m}(P_{p-y'-i}, \tau^{-k} M_{y'}) \\ &= \text{Hom}_{\mathcal{C}_m}(P_{p-y'-i}, M_{k+y'}) \\ &= \text{Hom}_{\mathcal{C}_m}(P_{p-y'-i}, S_{p-k-y'}) \neq 0 \end{aligned}$$

because $p - y' - k \equiv p - y' - i$.

- If $T_1 \in \{\tau^{-y'+kp} P_{p+j} | k \in \mathbb{Z}, j \in \{0, \dots, q\}\}$:

$$\begin{aligned} \text{Hom}_{\mathcal{C}_m}(\tau^{-y'+kp} P_{p+j}, M_{y'}) &\cong \text{Hom}_{\mathcal{C}_m}(P_{p+j}, \tau^{y'-kp} M_{y'}) \\ &= \text{Hom}_{\mathcal{C}_m}(P_{p+j}, M_{y'-y'+kp}) \\ &= \text{Hom}_{\mathcal{C}_m}(P_{p+j}, M_0) \neq 0 \end{aligned}$$

because there is a monomorphism $P_{p+j} \hookrightarrow M_0$.

We conclude that, in all the cases that we have seen above, there is a morphism $T_1 \rightarrow T_2$ in \mathcal{C}_m . Now, let us see the case where d_1 and d_2 are in different $(m+2)$ -gons. As before, since both share

a vertex of $P_{p,q,m}$ there must be a third diagonal d_3 having the same vertex in common. Then, we have arrows $d_1 \rightarrow d_3 \rightarrow d_2$ in Q_Δ^0 .

At first, suppose that d_3 and d_2 are diagonals of the same type sharing a vertex. Then, if $x := t(d_2)$, we also have $x = t(d_3)$ because, if we had $x = s(d_3)$, then it would be impossible for the diagonal d_3 to be in the middle of d_1 and d_2 . Therefore d_2 and d_3 share the final vertex and consequently $d_2 \in d_3^{ur}$. Assume that $F(d_1) = P_a$ (otherwise we apply τ as many times as necessary). Then, since there is a nonzero morphism $T_1 = P_a \rightarrow T_3$, by lemma 4.12 we obtain $T_3 \in M_*^{dl}$, where $*$ = 0 if $a \in \{p, p+1, \dots, p+q-1, 0\}$ or $*$ = $p-a$ if $a \in \{1, \dots, p-1\}$. Since $d_2 \in d_3^{ur}$ we have T_2, T_3 both lie in the same ray M_*^{dl} .

If $a \in \{1, \dots, p-1\}$ the ray M_*^{dl} is

$$\begin{array}{ccccccc} & 1 & 0 & & 1 & 0 & \\ & p+1 & & a & p+1 & & a \\ \cdots \rightarrow & \vdots & & \vdots & \vdots & & \vdots \\ & p+q-1 & & p-1 & p+q-1 & & p-1 \\ & & p & & & p & \\ & & & & & & \\ & & & a & & & \\ & & & a+1 & & & \\ & & \rightarrow & \vdots & \rightarrow \cdots \rightarrow & a+1 & \rightarrow a \\ & & & \vdots & & & \\ & & & p-1 & & & \end{array}$$

and it follows easily from this that the composition $P_a \rightarrow T_3 \rightarrow T_2$ is nonzero. Since, by remark 4.11, $\dim(\text{Hom}_{\mathbb{C}_m}(P_a, T_2)) = 1$ we deduce that the morphism $P_a \rightarrow T_2$ factors through T_3 . The computation is completely analogous if $a \in \{p, p+1, \dots, p+q-1, 0\}$.

Finally, suppose that d_3 is a diagonal of the same type as d_1 (type 1). We want to show that the morphism $T_1 \rightarrow T_2$ factors through T_3 . Since d_1 is a diagonal of type 1 we can assume that $T_1 = P$ with P projective in mod H . We know that there is a morphism $d_1 \rightarrow d_3$ then $d_3 \in d_1^{ur}$ and equivalently $T_3 \in T_1^{ur}$. If $T_1 = P_{p+z}$ with $z \in \{0, \dots, q-1\}$ we have that T_3 must be $P_{p+z'}$ with $0 < z' < z$. If instead, $T_1 = P_{p-y}$ with $y \in \{1, \dots, p\}$ we get that $T_3 = \tau^{-r} P_{p-y+r}$ with $r \in \{1, \dots, y-1\}$ or $T_3 = \tau^{-y} P_{p+x}$ with $x \in \{0, \dots, q-1\}$. See figure 1.

In all the cases the morphism $T_1 \rightarrow T_3$ is the inclusion of T_1 in T_3 , then if f is the nonzero morphism $T_3 \rightarrow T_2$, the composition $T_1 \rightarrow T_3 \rightarrow T_2$ is the restriction $f|_{T_1}$ which clearly is nonzero. Since, by remark 4.11, $\dim \operatorname{Hom}_{\mathcal{C}_m}(T_1, T_2) = 1$ we deduce that $f|_{T_1}$ is the morphism $T_1 \rightarrow T_2$ as we wanted to show. \square

6. THE QUIVER WITH RELATIONS ASSOCIATED TO AN $(m+2)$ -ANGULATION

In light of the previous results, we know how to find the ordinary quiver of an m -cluster tilted algebra of type $\tilde{\mathbb{A}}$. Now, our problem is to find the relations in such a quiver.

We start by defining an ideal I_Δ associated to an $(m+2)$ -angulation Δ of $P_{p,q,m}$ as follows:

6.1. Definition. Let i, j, k be the vertices in $(Q_\Delta^0)_0$ associated to the diagonals d_i, d_j and d_k respectively. Given consecutive arrows $i \xrightarrow{\alpha} j \xrightarrow{\beta} k$ in Q_Δ^0 we define the path $\beta\alpha$ to be zero if the diagonals d_i, d_j and d_k are in the same $(m+2)$ -gon in the $(m+2)$ -angulation Δ . Let I_Δ denote the ideal in the path algebra kQ_Δ^0 generated by all such relations.

Note that our definition agrees with the one given in [20] for the case \mathbb{A} .

6.2. Proposition. *The algebra kQ_Δ^0/I_Δ is isomorphic to the m -cluster tilted algebra $\text{End}_{\mathbf{e}_m}(F(\Delta))$.*

Proof. We first prove that all the relations in the ideal I_Δ are also relations in $\text{End}_{\mathcal{C}_m}(F(\Delta))$. Suppose that d_i , d_j and d_k are m -diagonals which are part of the same $(m+2)$ -gon arranged in such a way that d_i and d_j share a vertex of $P_{p,q,m}$, d_j and d_k share a vertex of $P_{p,q,m}$ but d_i , d_j and d_k have no common vertex.

By the definition of Q_Δ^0 we have arrows $d_i \xrightarrow{\alpha} d_j$ and $d_j \xrightarrow{\beta} d_k$ in Q_Δ^0 which are in bijection with the arrows $T_i \rightarrow T_j$ and $T_j \rightarrow T_k$ between the indecomposable factors of $T := F(\Delta)$. We want to show that the composition $T_i \rightarrow T_j \rightarrow T_k$ is zero in \mathcal{C}_m .

The proof will be divided into four cases, based on the type of the diagonals. Let us first observe that, by lemma 2.1, the three diagonals d_i , d_j and d_k cannot be simultaneously of type 1. Then, we have the following cases:

(a) The three diagonals are of type 2 (or 3): We have the following sub-cases.

1. The sources of d_j and d_k are the same: Since d_i , d_j and d_k are part of the same $(m+2)$ -gon, it is impossible to have $d_i \cap d_j = s(d_j) = s(d_k)$. Then $d_i \cap d_j = t(d_j)$. Since the sources of d_j et d_k coincide, the two diagonals are in the same tube (let us suppose T_p^r). Moreover, since there is a nonzero morphism $d_j \rightarrow d_k$ we have $d_k \in d_j^{dr}$ and then, by lemma 2.2, the diagonal d_i belongs to the tube T_p^{r-1} . Applying lemma 4.15 to the induced morphism $T_i \rightarrow T_j$ we obtain $T_j \in ((\mathcal{B}^-(\tau T_i)[1])^{dl})_{\leq \alpha(T_i)}$. If $\text{Hom}_{\mathcal{C}_m}(T_i, T_k) \neq 0$, by the same lemma, we see that T_k belongs to the same set as T_j . But, if this is the case, we cannot have $T_k \in T_j^{dr}$ as we saw above. Consequently, the composition $T_i \rightarrow T_j \rightarrow T_k$ is zero.
2. The target of d_k is equal to the source of d_j : Since there is a nonzero morphism $d_i \rightarrow d_j$ we have $t(d_j) = t(d_i)$ or $s(d_i) = t(d_j)$. In the first case, it is clear that the three diagonals cannot be part of the same $(m+2)$ -gon. Then $s(d_i) = t(d_j)$. Suppose that $d_k \in T_p^i$, then since $t(d_k) = s(d_j)$ lemma 2.2 implies $d_j \in T_p^{i-1}$. The same lemma applied to the diagonals d_i and d_j gives $d_i \in T_p^{i-2}$. We want to show that $\text{Hom}_{\mathcal{C}_m}(T_i, T_k) = 0$. If $m > 2$ let $T'_k \in \mathcal{T}_p^{i-2}$ be such that $T_k := T'_k[2]$, then $\text{Hom}_{\mathcal{C}_m}(T_i, T_k) = \text{Hom}_{\mathcal{C}_m}(T_i, T'_k[2]) = 0$ by lemma 3.1. If instead $m = 2$ the two diagonals d_i and d_k belong to the same tube, say T^0 , and d_j belongs to the tube T^1 . Thus, the only way to have a nonzero morphism $d_i \rightarrow d_k$ is that both diagonals share a vertex. That is, $s(d_i) = s(d_k)$ or $t(d_i) = t(d_k)$. Suppose $s(d_i) = s(d_k)$, then since $t(d_j) = s(d_i)$ we have $t(d_j) = s(d_k)$. Recall that we are in the case where $s(d_j) = t(d_k)$, in consequence the diagonals d_j and d_k share the same extremities and then it is not possible to have one diagonal d_i of the same type so that the three diagonals delimit the same $(m+2)$ -gon. If $t(d_i) = t(d_k)$ we conclude similarly that the diagonals d_j and d_i share the same extremities and thus it is not possible to have one diagonal d_k of the same type so that the three diagonals delimit the same $(m+2)$ -gon. Therefore it is not possible to have a nonzero morphism $T_i \rightarrow T_k$.
3. The targets of d_j and d_k are equal: Let us see the possibilities for d_i . If $t(d_i) = s(d_j)$ the three diagonals cannot be part of the same $(m+2)$ -gon. Then $s(d_i) = s(d_j)$ and consequently $T_j \in T_i^{dr}$. Since the three diagonals are in the same tube then by lemma 4.7, to prove that $\text{Hom}_{\mathcal{C}_m}(T_i, T_k) = 0$ it suffices to show that $T_k \notin T_i^{ur} \cup T_i^{dr}$. This is

due to the facts that $T_j \in T_i^{dr}$ and $t(d_j) = t(d_k)$ implies $T_k \in T_j^{ur}$.

(b) There is one diagonal of each type: Since $d_i \cap d_j$ is a vertex in the inner (or outer) polygon and $d_j \cap d_k$ is a vertex in the outer (or inner, respectively) polygon, the diagonal d_j must be the diagonal of type 1. Assume $d_j \in S^z$, $d_i \in T_p^x$ is the diagonal of type 2 and $d_k \in T_q^y$ is the diagonal of type 3. Since there are arrows $d_i \rightarrow d_j \rightarrow d_k$ we have $z = y = x + 1$. Then, if we take $T_{k'} \in \mathcal{T}_p^x$ such that $T_k = T_{k'}[1]$, by lemma 3.2 we obtain $\text{Hom}_{\mathcal{C}_m}(T_i, T_k) = \text{Hom}_{\mathcal{C}_m}(T_i, T_{k'}[1]) = 0$.

(c) Two diagonals are of type 1 and the other one is of type 2 (or 3): We have the following sub-cases.

1. d_i and d_j are of type 1 and d_k is of type 2 (or 3): Let $d_i \cap d_j = \{x\}$. Assume x belongs to the outer polygon. The existence of a morphism $d_j \rightarrow d_k$ with d_k a diagonal of type 2, implies $d_j \cap d_k = \{y\}$ with y also in the outer polygon. Since d_j is a diagonal of type 1, it has just one vertex in the outer polygon, and thus $x = y$ and the vertex x is the source or the target of d_k . Consequently $d_i \cap d_j \cap d_k = \{x\}$ and the three diagonals cannot be part of the same $(m+2)$ -gon. Therefore x belongs to the inner polygon, both diagonals of type 1 have target x and the diagonals d_j and d_k share the vertex y from the outer polygon. Hence $d_j \in d_i^{dr}$, that is, $T_j \in T_i^{dr}$. Without loss of generality, since T_i belongs to the transjective component S^0 we can assume that $T_i = P[0]$ ($= P$ by abuse of notation) with P projective in $\text{mod } H$ (for, otherwise, we apply τ as many times as necessary to T_i until we find a projective). Then, $T_i \in P_p^{ur} \cup P_p^{dr}$ where P_p is the projective associated with the vertex p of the quiver.

If $T_i \in P_p^{dr}$ then $T_i = P_a$ with $a \in \{1, \dots, p\}$ and $T_j = P_b$ with $0 \leq b < a < p$. Since there is a nonzero morphism $T_j \rightarrow T_k$, lemma 4.12 implies $T_k \in S_b^{dl}$. If the composition $T_i \rightarrow T_k$ is not zero, the same lemma applied to T_i gives $T_k \in S_a^{dl}$. Clearly $S_a^{dl} \neq S_b^{dl}$ and consequently $\text{Hom}_{\mathcal{C}_m}(T_i, T_k) = 0$.

If instead $T_i \in P_p^{ur} \setminus \{P_p\}$, $T_i = P_{p+z}$ (with $z \in \{1, \dots, q-1\}$). As $T_j \in T_i^{dr}$ we have $T_j = \tau^{-r} P_{p+z+r}$ with $r > 0$ and $1 \leq z \leq q-1$. If the composition $T_i \rightarrow T_j \rightarrow T_k$ is not zero, then by lemma 4.12, we obtain $T_k \in M_0^{dl}$. Since $0 \neq \text{Hom}_{\mathcal{C}_m}(T_j, T_k) = \text{Hom}_{\mathcal{C}_m}(\tau^{-r} P_{p+z+r}, T_k) \cong \text{Hom}_{\mathcal{C}_m}(P_{p+z+r}, \tau^r T_k)$ the same lemma gives $T_k \in \tau^{-r} M_0^{dl} = M_r^{dl}$. Then, $M_r^{dl} = M_0^{dl}$ which implies $r \equiv 0 \pmod{p}$ and equivalently $r = kp$ with $k \in \mathbb{Z}$. The diagonal d_i is in bijection with the projective P_{p+z} and the diagonal d_j is in bijection with $\tau^{-r} P_{p+z+r}$, then d_i is a diagonal between the vertices O_0 and I_x and $d_j = \tau^{-r} d$ with d a diagonal between the vertices O_0 and I_y and $y > x$. Then $d_j = \tau^{-r} d = d[-rm] = d[-kpm]$ and consequently d_j is a diagonal between the vertices O_0 and I_y and it is not possible to have a diagonal d_k of type 2 between both.

2. d_i is of type 2 (or 3) and d_k and d_j are of type 1: It is the case dual to the one considered in 1 above.

3. d_i and d_k are both of type 1 and d_j is of type 2 (or 3): Suppose $d_i \cap d_j = \{x\}$ and $d_j \cap d_k = \{y\}$ with x, y in the outer polygon in such a way that $s(d_j) = y$ and $t(d_j) = x$. In particular $x \neq y$ and $x = y + km + 1 \pmod{mp}$ with $k \geq 1$. Assume $d_i \in S^r$, then

$x \equiv m - r \pmod{m}$ and hence $y \equiv x - 1 \equiv m - (r + 1) \pmod{m}$. Thus $d_k \in S^{r+1}$ and $d_j \in T_p^r$. Since $d_i \in S^r$ and $d_k \in S^{r+1}$ there are $s, s' \in \mathbb{Z}$ and P, P' projective objects (in $\text{mod } H$) such that $T_i = \tau^s P[r]$ and $T_k = \tau^{s'} P'[r + 1]$. Then:

$$\begin{aligned} \text{Hom}_{\mathcal{C}_m}(T_i, T_k) &= \text{Hom}_{\mathcal{C}_m}(\tau^s P[r], \tau^{s'} P'[r + 1]) \\ &\cong \text{Hom}_{\mathcal{C}_m}(\tau^s P, \tau^{s'} P'[1]) \\ &= \text{Hom}_{\mathcal{C}_m}(P, \tau^{s-s'} P'[1]) = 0 \end{aligned}$$

by lemma 4.8.

- (d) Two diagonals are of type 2 (or 3) and the third one is of type 1: First of all, let us see that d_j cannot be the diagonal of type 1. If this is the case, since there are arrows $d_i \rightarrow d_j$ and $d_j \rightarrow d_k$ the three diagonals have to share a vertex of the outer polygon (inner if d_i, d_k are of type 3), but in this situation it is not possible that the three diagonals delimit the same $(m + 2)$ -gon. Then, assume d_i and d_j are of type 2 and d_k is of type 1. If $s(d_i) = s(d_j)$ the possible diagonals d_k of type 1 either have an intersection with $d_i \cup d_j$ or do not delimit the same $(m + 2)$ -gon. (See figure 2).

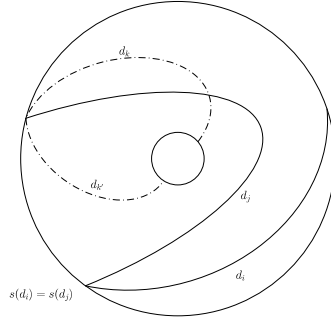


FIGURE 2. The possible d_k if $s(d_i) = s(d_j)$.

Hence, $t(d_j) = s(d_i)$ and if $d_j \in T^r$ then $d_i \in T^{r-1}$ by lemma 2.2. In addition $d_k \in S^{r+1}$. If $m > 2$ let $d'_k \in S^{r-1}$ be a diagonal such that $d_k = d'_k[2]$. Consequently, lemma 3.1 implies $\text{Hom}_{\mathcal{C}_m}(T_i, T_k) \cong \text{Hom}_{\mathcal{C}_m}(T_i, T'_k[2]) = 0$. If $m = 2$ we can assume $d_i \in T^0$, $d_j \in T^1$ and $d_k \in S^0$. Thus

$$\text{Hom}_{\mathcal{C}_m}(T_i, T_k) = \bigoplus_{j \in \mathbb{Z}} \text{Hom}_{D^b(H)}(T_i, F^j T_k) = \bigoplus_{j \in \mathbb{Z}} \text{Hom}_{D^b(H)}(T_i, \tau^{-j} T_k[2j])$$

All terms $\text{Hom}_{D^b(H)}(T_i, \tau^{-j} T_k[2j])$ with $j < 0$ are zero because the objects T_i and $\tau^{-j} T_k$ are in $(\text{mod } H)[0]$ and there are no backward morphisms in the derived category $D^b(\text{mod } H)$.

H). If $2j \geq 2$ all terms $\text{Hom}_{D^b(H)}(T_i, \tau^{-j}T_k[2j])$ are zero since H is hereditary. Finally, if $j = 0$ the term $\text{Hom}_{D^b(H)}(T_i, T_k) = 0$ because there are no morphisms between an object in the tube \mathcal{T}^0 and an object in the component \mathcal{S}^0 . In consequence, $\text{Hom}_{\mathcal{C}_m}(T_i, T_k) = 0$.

So far we have seen that all the relations in the ideal I_Δ are also relations in $\text{End}_{\mathcal{C}_m}(F(\Delta))$. To finish, we still have to prove that there are no other possible relations in $\text{End}_{\mathcal{C}_m}(F(\Delta))$.

We begin by proving there are not relations of (minimal) length greater than or equal to three. Let $(A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D)$ be a relation of length $r \geq 3$, where $\gamma\beta\alpha = 0$. The arrows $(A \rightarrow B)$ and $(C \rightarrow D)$ are of length 1 and the path $(B \rightsquigarrow C)$ can have arbitrary length. In addition, suppose that the compositions $(A \rightarrow B \rightsquigarrow C)$ and $(B \rightsquigarrow C \rightarrow D)$ are nonzero. Let d_* be the m -diagonal such that $F(d_*) = *$ with $*$ in $\{A, B, C, D\}$. We divide the proof into 5 cases.

- (a') The 4 diagonals are of type 1. By lemma 4.8 there are no nonzero morphisms between the components \mathcal{S}^d and \mathcal{S}^{d+1} . In consequence the four diagonals must be in the same component \mathcal{S}^d and thus in the ray d_A^{ur} or the ray d_A^{dr} . Hence, the composition $(A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D)$ cannot be zero, contrary to our assumption.
- (b') Three diagonals are of type 1 and one is of type 2 (or 3). By the argument given in the case above, the three diagonals of type 1 have to be in the same component \mathcal{S}^d . Thus, we can assume that d_A, d_B and d_C are of type 1 in \mathcal{S}^0 and d_D is of type 2 in a tube. Since the morphism $(C \rightarrow D)$ is not zero, the diagonal d_D shares with d_C the vertex in the outer polygon. In consequence, $d_D \in T^0$ and all objects are in the same degree 0. Hence, we can think like in the case $m = 1$ and since the 1-cluster tilted algebras of type \tilde{A} are gentle, we cannot have the relation $(A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} D) = 0$.

- (c') Two diagonals are of type 2 (or 3) and two are of type 1. Lemma 4.8 implies both diagonals of type 1 are consecutive in the same component \mathcal{S}^d or one diagonal is d_A and the other one is d_D .

To begin, suppose we are in the first case. Then, we have the following sub-cases:

1. If d_A and d_B are in \mathcal{S}^0 and d_C and d_D are in the same tube T^0 , we can argue as in the case $m = 1$ and conclude that it is not possible. If instead, d_C is in the tube T^0 and d_D in the tube T^1 , the calculation done in (d) above shows that $(B \xrightarrow{\beta} C \xrightarrow{\gamma} D) = 0$, a contradiction.
2. If d_C and d_D are in \mathcal{S}^1 , we have the case dual to the one considered above.
3. If d_B and d_C are in \mathcal{S}^0 the diagonal d_D has to be in T^1 . Hence, the computation done in (c) above implies that $(B \xrightarrow{\beta} C \xrightarrow{\gamma} D) = 0$, contrary to our assumption.

Finally, suppose that d_A and d_D are the diagonals of type 1. Then, we have two sub-cases:

1. If d_B and d_C are in the same tube we can assume that the objects B, C belong to the tube \mathcal{T}^d and the object D belongs to the component \mathcal{S}^{d+1} . Suppose that $A = P_a$ and

$D = \tau^r P_b[1]$. If $(A \rightarrow B \rightsquigarrow C) = 0$ we have a contradiction; otherwise, by lemma 4.12 we deduce that $B, C \in M_*^{dl}$ with

$$* = \begin{cases} p - a, & \text{if } a \in \{1, \dots, p-1\}; \\ 0, & \text{otherwise.} \end{cases}$$

Since $(C \rightarrow D) \neq 0$ lemma 4.13 implies $C \in \tau^{r-1} M_{\sharp}^{dr} = M_{\sharp+r-1}^{dr}$ with

$$\sharp = \begin{cases} p - b, & \text{if } b \in \{1, \dots, p-1\}; \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\{C\} = M_{\sharp+r-1}^{dr} \cap M_*^{dl}$. If $(B \rightarrow D)$ is not zero we would have $B \in M_{\sharp+r-1}^{dr} \cap M_*^{dl} = \{C\}$, which is impossible.

2. If d_B and d_C are in different tubes, the computation done in (d) above shows that $(B \xrightarrow{\beta} C \xrightarrow{\gamma} D) = 0$, a contradiction.

(d') Three diagonals are of type 2 (or 3) and one is of type 1 . We have the following sub-cases.

1. Suppose that d_A and d_B are in T^0 , d_C is in S^1 and d_D in T^1 . Then we can assume that $C = \tau^r P_c[1]$ and since $(A \rightarrow B \rightarrow C) \neq 0$, lemma 4.13 implies $A, B \in M_{*-1+r}^{dr}$ with

$$* = \begin{cases} p - c, & \text{if } c \in \{1, \dots, p-1\}; \\ 0, & \text{otherwise.} \end{cases}$$

If $(B \rightsquigarrow C \rightarrow D) \neq 0$, then the level of B is not $p-1$ and thus $\alpha(B) \neq 0$. Lemma 4.15 now yields $D \in (\mathcal{B}^-(\tau B)[1])^{dl} = (\tau \mathcal{B}^-(B)[1])^{dl} = (\tau M_{*-1+r}[1])^{dl} = (M_{*-1+r-1}[1])^{dl}$. On the other hand, since $(C \rightarrow D) \neq 0$, we have $\tau^{-r} D \in (M_*[1])^{dl}$ and consequently $D \in (M_{*+r}[1])^{dl} \cap (M_{*+r-2}[1])^{dl} = \emptyset$ except if $p = 2$. But, for $p = 2$, this is not possible, because we cannot have two factors of T in the mouth of the tube.

2. Assume that d_A and d_B are in T^0 , d_C in T^1 and d_D in S^2 . Since the morphism $(C \rightarrow D)$ is not zero, the diagonals d_D and d_C share the source of d_C . Thus, the calculation done in (d) above gives $(B \xrightarrow{\beta} C \xrightarrow{\gamma} D) = 0$, a contradiction.
3. Assume that the three diagonals of type 2 are one in the tube T^d and the others two in the tube T^{d+1} . Then, the computation done in (a) – 1 above gives that the composition of the morphisms between these three diagonals is zero, a contradiction.
4. Suppose that each diagonal of type 2 is in a different tube. Then, for example, $d_A \in T^0$, $d_B \in T^1$ and $d_C \in T^2$. Now the calculation done in (a) – 2, shows that $(A \rightarrow B \rightsquigarrow C) = 0$, which contradicts our assumption.
5. Suppose that d_A belongs to T^0 , d_B belongs to T^1 , d_C to S^2 and d_D to T^2 . Then, the three diagonals d_A, d_B and d_C are in the situation of the case (d) above and, in consequence, the composition $(A \rightarrow B \rightsquigarrow C) = 0$, a contradiction.
6. Suppose that the three diagonals d_A, d_B and d_C are in the tube T^0 and d_D is in the component S^1 . Then, we can think like in the case $m = 1$ because all the diagonals

are in the same degree. Since we know that for $m = 1$ there are not relations of length greater than or equal to 3 we finish.

(e') The four diagonals are of type 2 (or 3). Let us see the possibles sub-cases which are not among the ones considered in (d') for the three diagonals of type 2.

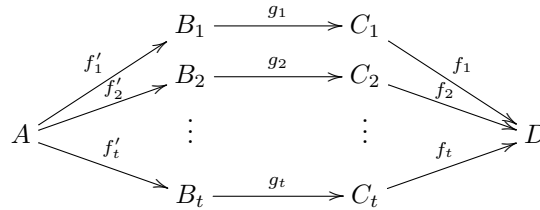
1. d_A and d_B are in the tube T^0 and d_C and d_D are in the tube T^1 . Then, we have two options for the objects C and D . Either $D \in C^{ur}$, or $D \in C^{dr}$. Suppose that $D \in C^{ur}$. Let C' and D' be objects such that $C'[1] = C$ and $D'[1] = D$, then $D' \in C'^{ur}$. Since $(A \rightarrow C)$ and $(B \rightarrow D)$ are nonzero, the $(m+1)$ -CY property of \mathcal{C}_m gives that $0 \neq \text{Hom}_{\mathcal{C}_m}(A, C'[1]) \cong D \text{Hom}_{\mathcal{C}_m}(C', \tau A)$ and $0 \neq \text{Hom}_{\mathcal{C}_m}(B, D'[1]) \cong D \text{Hom}_{\mathcal{C}_m}(D', \tau B)$. Thus, $C' \in (\tau A)^-$ and $D' \in (\tau B)^-$. The same computation applied to the zero morphism $(A \rightarrow D)$ gives $D' \in ((\tau A)^-)^c$. On the other hand, since there is a nonzero morphism $(A \rightarrow B)$ the sources of d_A and d_B must be the same vertex. Hence, $B \in A^{dr}$ and equivalently $\tau B \in (\tau A)^{dr}$. In consequence, $D' \in C'^{ur} \cap ((\tau B)^- \setminus (\tau A)^-) = \emptyset$, which is not possible. To conclude, assume that $D \in C^{dr}$, then $s(d_C) = s(d_D)$ and the computation done in the previous part (a) yields that $(B \rightsquigarrow C \rightarrow D) = 0$.

2. d_A, d_B and d_C are in the tube T^0 and d_D is in the tube T^1 . Note that A, B and C are in the same ray, that is, the ray C^{dl} or A^{dr} . If A, B and C belong to the ray C^{dl} then $\mathcal{B}^-(\tau C) \neq \mathcal{B}^-(\tau B)$ and since $(C \rightarrow D) \neq 0$, $D \in (\mathcal{B}^-(\tau C)[1])^{dl} \neq (\mathcal{B}^-(\tau B)[1])^{dl}$. In consequence, lemma 4.15 implies that $(B \rightarrow D) = 0$.

If instead A, B and C belong to the ray A^{dr} we have that $\mathcal{B}^-(\tau C) = \mathcal{B}^-(\tau A)$. In addition, since we have the morphisms $A \rightarrow B \rightarrow C$ in the same ray the level of A cannot be $p-1$, and then $\alpha(C) < \alpha(A) \neq 0$. Since $(C \rightarrow D) \neq 0$ we have $D \in ((\mathcal{B}^-(\tau C)[1])^{dl})_{\leq \alpha(C)} \subset ((\mathcal{B}^-(\tau A)[1])^{dl})_{\leq \alpha(A)}$. In consequence, lemma 4.15 implies that $(A \rightsquigarrow D) \neq 0$, a contradiction.

Note that in all cases except the case (a'), the two last sub-cases of (c') and the sub-case (e')-2 we did not really have to use that the composition $A \rightarrow B \rightsquigarrow C \rightarrow D$ is zero and we have actually proved that one of the compositions $A \rightarrow B \rightsquigarrow C$ or $B \rightsquigarrow C \rightarrow D$ is zero.

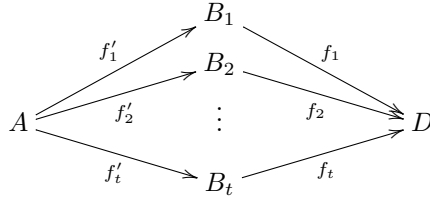
Next, we claim that there are no relations of the form



where a linear combination of compositions $f_i g_i f'_i$ is 0, $t \geq 2$ and the paths g_i can have arbitrary length. Since the idea is to prove that all the compositions $f_i g_i f'_i$ except at most one are zero, we can assume $t = 2$. The last remark implies that if the diagonals d_A, d_{B_1}, d_{C_1} and d_D are not in the three cases considered above we cannot have such a relation. Then, it just remains to see these three cases.

- (i) Case (a'). In this case the diagonals d_A, d_{B_1}, d_{C_1} and d_D are of type 1. By lemma 4.8 every morphism from S^d to S^{d+1} is zero. In consequence the four diagonals have to be in the same component S^d and so in the ray d_A^{ur} or the ray d_A^{dr} . The same argument applied to the diagonals d_A, d_{B_2}, d_{C_2} and d_D implies that the four diagonals have to be in the ray d_A^{ur} or the ray d_A^{dr} . Thus, D must be the last object in the two rays A^{ur} and A^{dr} and we can assume that $A = P_p$ and $D = P_0$. However, if this is the case then we cannot have the required relation.
- (ii) The two last sub-cases of case (c'). Since the composition $A \rightarrow D$ cannot be zero, we do not have these sub-cases.
- (iii) Case (e')-2. In this case d_A, d_{B_1} and d_{C_1} are in the tube T^0 and d_D is in the tube T^1 . We can assume that A, B_1 and C_1 are in the ray A^{ur} . Then, B_2 and C_2 have to lie on the ray A^{dr} . Hence, the computation done in (e')-2 shows that the composition $B_1 \rightarrow D$ is zero, which is impossible.

Finally, there only remains to see that we cannot have relations of the form



where the sum of the compositions $f_i f'_i$ has to be 0 and $t \geq 2$. The idea is to show that one of the compositions $f_i f'_i$ has to be zero. Then, we can assume that $t = 2$.

At first, we observe that if the four factors A, B_1, B_2 and D of $F(\Delta)$ are in the same component S^i the computation done in (i) implies that we cannot have a commutativity relation. Next, observe that if $A \in S^0$ and $B_1, B_2, D \in \mathcal{T}_p^0$ the three objects B_1, B_2, D have to be in the same ray and consequently the endomorphisms algebra of $F(\Delta)$ does not contains the required relation. Finally, since every morphism from S^0 to S^1 is zero, the only possibility is to have $A, B_1, B_2 \in S^0$ and $D \in \mathcal{T}_p^0$. But then we can think like in the case $m = 1$ where we cannot have such a relation. See [20, Proposition 3.1].

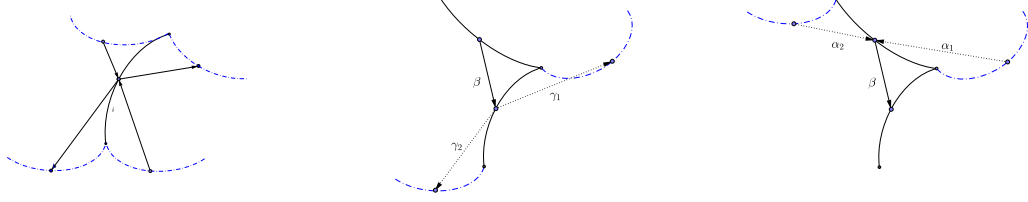
Summarizing, we have seen that the only possible relations in the algebra $\text{End}_{\mathcal{C}_m}(F(\Delta))$ are the relations given by the ideal I_Δ . \square

7. THE m -CLUSTER TILTED ALGEBRAS OF TYPE $\tilde{\mathbb{A}}$

The aim of this section is to show that m -cluster tilted algebras of type $\tilde{\mathbb{A}}$ are gentle and to find a characterization of their bound quivers. Murphy in [20] showed that m -cluster tilted algebras of type \mathbb{A} are gentle for any $m \geq 1$. We want to prove the same for the type $\tilde{\mathbb{A}}$. For $m = 1$ the result was showed in [1]. For any $m \geq 2$ is a easily consequence of proposition 6.2.

7.1. Proposition. *The m -cluster-tilted algebras of type $\tilde{\mathbb{A}}$ are gentle for any $m \geq 2$.*

Proof. The result follows from considerations of the possible divisions of $P_{p,q,m}$. The following figures make the required properties clear.



Finally observe that by definition, the ideal I_Δ is generated by monomial quadratic relations. \square

Given a bound quiver (Q, I) and an integer m , a cycle is called *m-saturated* if it is an oriented cycle consisting of $m + 2$ arrows such that the composition of any two consecutive arrows on this cycle belongs to I . Recall that two relations r and r' in the bound quiver (Q, I) are said to be *consecutive* if there is a walk $v = wr = r'w'$ in (Q, I) such that r and r' point in the same direction and share an arrow.

For the following definition we fix a natural $m \geq 2$.

7.2. Definition. Let $\tilde{\mathcal{C}}$ be a cycle without relations (oriented or not) and fix an orientation of its arrows. We say that $A \cong kQ/I$ is an *algebra with root $\tilde{\mathcal{C}}$* if its bound quiver can be constructed as follows:

- (1) We add to the cycle $\tilde{\mathcal{C}}$ gentle quivers in such a way that the final quiver remains gentle and connected. These gentle quivers that we add can have cycles all of which are *m-saturated*. We call these quivers *rays*.
- (2) We can add relations to the cycle $\tilde{\mathcal{C}}$. If the cycle $\tilde{\mathcal{C}}$ is oriented we must add at least one relation.

Also, we will refer to the cycle $\tilde{\mathcal{C}}$ as the *root cycle*.

In the sequel let $\tilde{\mathcal{C}}$ be a non-saturated cycle and $A \cong kQ/I$ an algebra with root $\tilde{\mathcal{C}}$.

7.3. Definition. Let c be a vertex (which is not in an *m-saturated* cycle) in a ray. We said that c is the *union vertex* of the ray, if c lies also in the root cycle.

7.4. Remarks. Let $\tilde{\mathcal{C}}$ be a cycle and A an algebra with root $\tilde{\mathcal{C}}$.

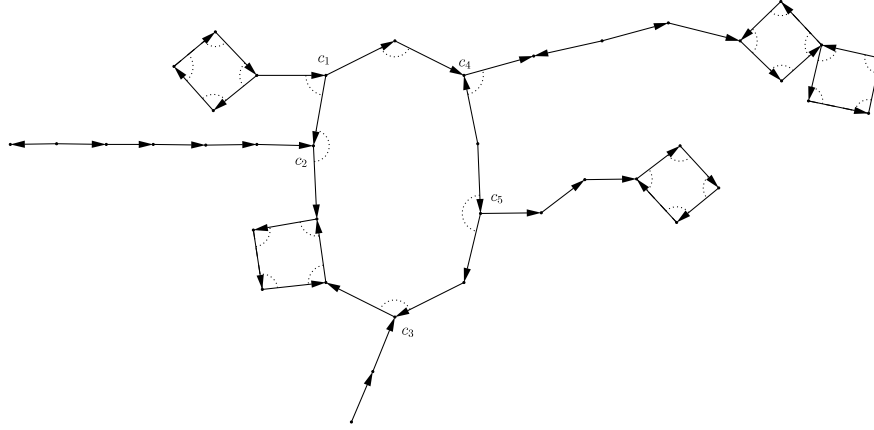
- (1) Each ray of A can share with the cycle $\tilde{\mathcal{C}}$ at most $m + 2$ vertices. If it shares just one vertex, this vertex is the union vertex of the ray. If it shares more than one vertex, the ray and the cycle $\tilde{\mathcal{C}}$ are connected through an *m-saturated* cycle.
- (2) For each union vertex there is at least one relation involving at least one arrow of $\tilde{\mathcal{C}}$.

7.5. Definition. Let a be an union vertex and let ρ be the involved relation in the root cycle. The relation ρ is called:

- a) *internal union relation* of the ray if both arrows of the relation belong to the root cycle.
- b) *external union relation* of the ray if just one arrow of the relation belongs to the root cycle.

Now, we assign an orientation to each union relation. Let ρ be an union relation. We say that ρ is *clockwise* (or *counterclockwise*) if the involved arrows which lie in the root cycle are clockwise (or counterclockwise, respectively) oriented.

7.6. **Example.** Let A be the algebra given by the following bound quiver.



The relations at the vertices c_1 and c_4 are counterclockwise external union relations. The one at the vertex c_2 is a counterclockwise internal union relation. Finally, the relations at the vertices c_3 and c_5 are clockwise internal union relations.

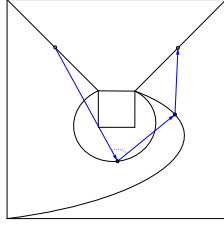
We collect the following easy consequences of proposition 6.2.

7.7. **Remarks.** Let $A \cong kQ/I$ be a connected m -cluster tilted algebra of type \tilde{A} , then:

- (1) (Q, I) does not contain non-saturated cycles or else has exactly one non-saturated cycle \tilde{C} in such a way that A is an algebra with root \tilde{C} .
- (2) In the first case the only possible cycles are m -saturated.
- (3) If $m \neq 1$, we can have relations outside of m -saturated cycles, but with the following restriction: we can have at most $m - 1$ consecutive relations outside of an m -saturated cycle .

In particular the previous remark implies that we do not have uniqueness of type for m -cluster tilted algebras because every m -cluster tilted algebra of type \tilde{A} without a non-saturated cycle is at the same time an m -cluster tilted algebras of type A . The following example illustrate this remark.

7.8. **Example.** Let (Q, I) be a bound quiver associated to the following 4-angulation Δ of $P_{2,2,2}$.



Then the algebra kQ/I is a 2-cluster tilted algebra of type $\tilde{A}_{2,2}$ and also of type A_4 .

Since m -cluster tilted algebras of type A are well understood, [20, 8, 13] we are going to focus in m -cluster tilted algebras of type \tilde{A} having at least one non-saturated cycle and so having (at least) a root cycle. In this case we choose one and we fix it. We start with the relations in the root cycle.

7.9. Definition. Let ρ be a relation in the root cycle. The relation ρ is said *strictly internal* if the two arrows of the relation belong to the root cycle, but not to any m -saturated cycle. In addition, we say that the strictly internal relation is *clockwise* (or *counterclockwise*) if both involved arrows are clockwise (or counterclockwise, respectively) oriented.

Let α_h be the number of clockwise strictly internal relations and α_a the number of counterclockwise strictly internal relations.

7.10. Remark. Let \mathcal{C} be an m -saturated cycle sharing at least two vertices i and j with the root cycle. Then, the root cycle orientation induces an orientation on the arrows of \mathcal{C} . That is that there are k arrows between the vertices i and j clockwise oriented and $m + 2 - k$ arrows counterclockwise oriented.

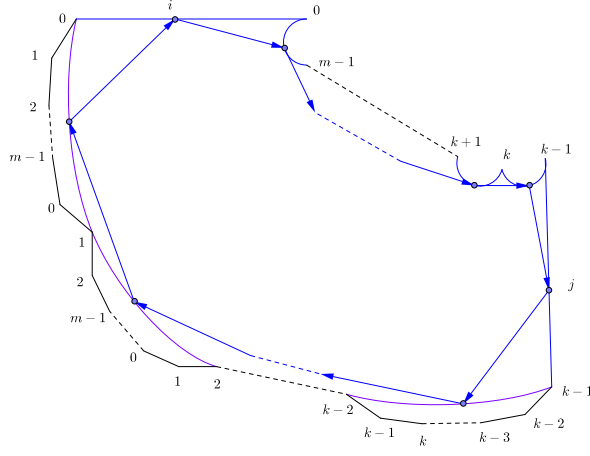
Given an m -saturated cycle \mathcal{C} having at least two vertices in common with the root cycle denote by $\beta_a(\mathcal{C})$ the number of arrows of \mathcal{C} with counterclockwise orientation less one and $\beta_h(\mathcal{C})$ the number of arrows of \mathcal{C} with clockwise orientation less one.

Observe that for any such cycle \mathcal{C} , we have $\beta_a(\mathcal{C}) + \beta_h(\mathcal{C}) = m$.

7.11. Lemma. *Let \mathcal{C} be an m -saturated cycle as in the above remark. Then at least one of the following conditions hold.*

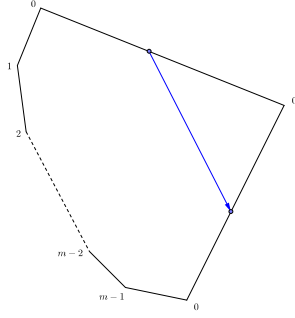
- (1) *There are at least $k - 1 = \beta_h(\mathcal{C})$ strictly internal counterclockwise relations in the root cycle.*
- (2) *There are at least $m - k + 1 = \beta_a(\mathcal{C})$ strictly internal clockwise relations in the root cycle.*
- (3) *There is another m -saturated cycle \mathcal{C}' with $\beta_h(\mathcal{C}') = \beta_a(\mathcal{C})$ and $\beta_a(\mathcal{C}') = \beta_h(\mathcal{C})$.*

Proof. Suppose that we have a cycle \mathcal{C} with k arrows between the vertices i and j clockwise oriented and $m + 2 - k$ arrows counterclockwise oriented. Then, in the corresponding $(m + 2)$ -angulation we have the following situation where to simplify the notation we label the vertices of the outer and inner polygon modulo m .

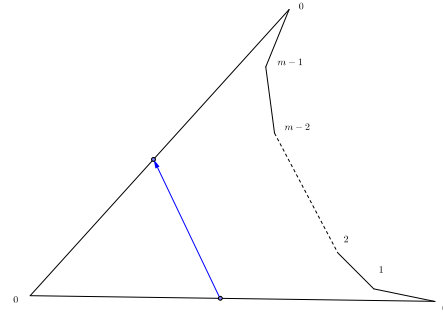


If we want to have a non-saturated cycle involving the vertices i and j , we must find a partition from the region of $P_{p,q,m}$ delimited by the diagonal i , the edge of the outer polygon lying between the vertices 0 and $k-1$ clockwise, the diagonal j and the edge of the inner polygon lying between the vertices $k-1$ and 0 counterclockwise.

If we add an arrow to the non-saturated cycle we have the following sub-angulations.

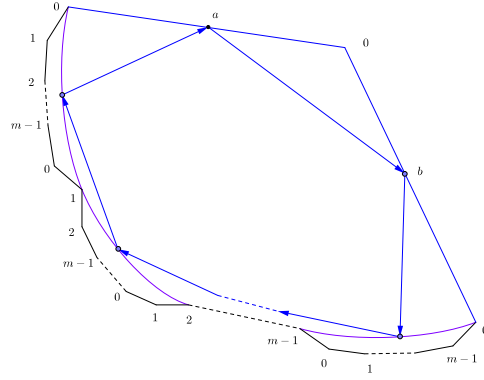


counterclockwise arrow

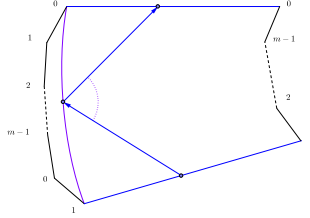


clockwise arrow

Then, when we add arrows the number modulo m of vertices of the outer and inner polygon do not change. We have the same situation when we add an m -saturated cycle sharing just one arrow with the root cycle (the arrow between the vertices a and b). See the figure below.

counterclockwise m -saturated cycle

If we add a clockwise relation we have to add one to the value of the outer polygon vertex and subtract one to the value of the inner polygon vertex. See the following figure.



clockwise relation

We have the dual case if we add one counterclockwise relation. That is, we subtract one to the value of the outer polygon vertex and we add one to the value of the inner polygon vertex.

Then, if we add $m - k + 1$ clockwise relations the vertex of the outer polygon is going to take the value $k - 1 + m - k + 1 = m \equiv 0$ (modulo m), and the vertex of the inner polygon is going to take the value $k - 1 - (m - k + 1) \equiv k - 1 - (k - 1) \equiv 0$ (modulo m).

It follows from these considerations that the only ways to "close" the non-saturated (root) cycle which contains the vertices i and j is to add $m - k + 1$ internal relations in the clockwise sense, or $k - 1$ relations in the counterclockwise sense, or another m -saturated cycle with k arrows in the counterclockwise sense and $m - k + 2$ arrows in the clockwise sense. \square

The lemma above allows us to make the following definition.

7.12. Definition. Let \mathcal{C} be an m -saturated cycle sharing at least two vertices with the root cycle. We say that the cycle \mathcal{C} is *clockwise* (or *counterclockwise*) if the condition 1 (or 2, respectively) holds. If 3 holds or 1 and 2 simultaneously hold, we say that \mathcal{C} is clockwise if $\beta_h(\mathcal{C}) \leq \beta_a(\mathcal{C})$ or counterclockwise otherwise.

We fix the following notation, let \mathfrak{C}_h be the set of clockwise m -saturated cycles and \mathfrak{C}_a the set of counterclockwise m -saturated cycles.

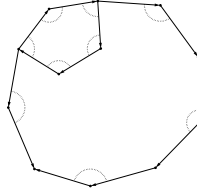
7.13. Definition. We define the *number of clockwise internal relations* r_h as

$$r_h = \alpha_h + \sum_{\mathcal{C} \in \mathfrak{C}_h} \beta_h(\mathcal{C})$$

In the same way we define the number of counterclockwise internal relations r_a by

$$r_a = \alpha_a + \sum_{\mathcal{C} \in \mathfrak{C}_a} \beta_a(\mathcal{C})$$

7.14. Example. Let \mathcal{C} be the 3-saturated cycle of the following bound quiver and fix as root cycle the cycle of length 10.



Then, $\beta_h(\mathcal{C}) = 1$ and $\beta_a(\mathcal{C}) = 2$. Since $\alpha_h = 3 > \beta_a(\mathcal{C})$, $\alpha_a = 1 = \beta_h(\mathcal{C})$ and $\beta_h(\mathcal{C}) \leq \beta_a(\mathcal{C})$; the cycle \mathcal{C} is counterclockwise.

The number of clockwise internal relations is $r_h = 3$ and the number of counterclockwise internal relations is $r_a = 1 + 2 = 3$. Then $r_h \equiv r_a$ modulo $m = 3$.

In general, we have the following property.

7.15. Proposition. Let A be a connected m -cluster tilted algebra of type $\tilde{\mathbb{A}}$ with a root cycle. If there are internal relations on the root cycle, then the number modulo m of clockwise internal relations is equal to the number of counterclockwise internal relations. That is, $r_h \equiv r_a$ modulo m .

Proof. To begin, assume that the set $\mathfrak{C}_a \cup \mathfrak{C}_h$ is empty. Then, the discussion in lemma 7.11 says that each time we add a clockwise oriented relation, we must in order to close the root cycle, add also a counterclockwise oriented relation. By the same lemma it also follows that if we add a multiple of m relations in any sense, then the values of the vertices of the outer or inner polygons do not change. Now, assume there is a cycle \mathcal{C} in \mathfrak{C}_a . Then, there are $\beta_h(\mathcal{C})$ counterclockwise

strictly internal relations. Therefore, $\alpha_a = \alpha'_a + \beta_h(\mathcal{C})$ with $\alpha'_a \equiv \alpha_h$ modulo m . In consequence, $r_a = \alpha'_a + \beta_h(\mathcal{C})$ and $r_h = \alpha_h + \beta_h(\mathcal{C})$. Clearly $r_h \equiv r_a$ modulo m . Similar considerations apply if we have an arbitrary number of clockwise or counterclockwise m -saturated cycles. \square

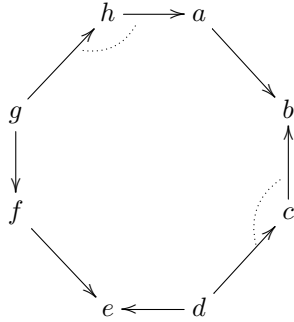
Finally, summarizing the results obtained about the bound quiver of an m -cluster tilted algebra of type $\tilde{\mathbb{A}}$ we get our main theorem.

7.16. Theorem. *Let (Q, I) be a connected bound quiver. Then (Q, I) is the bound quiver of a connected component of an m -cluster tilted algebra A of type $\tilde{\mathbb{A}}$ if and only if (Q, I) is a gentle quiver such that:*

- (a) *It can contain a non-saturated cycle $\tilde{\mathcal{C}}$ in such a way that A is an algebra with root $\tilde{\mathcal{C}}$.*
- (b) *If it contains more cycles, then all of them are m -saturated cycles.*
- (c) *Outside of an m -saturated cycle it can have at most $m - 1$ consecutive relations.*
- (d) *If $\tilde{\mathcal{C}}$ is an oriented cycle, then it must have at least one internal relation.*
- (e) *If there are internal relations in the root cycle, then the number of clockwise oriented relations is equal modulo m to the number of counterclockwise oriented.*

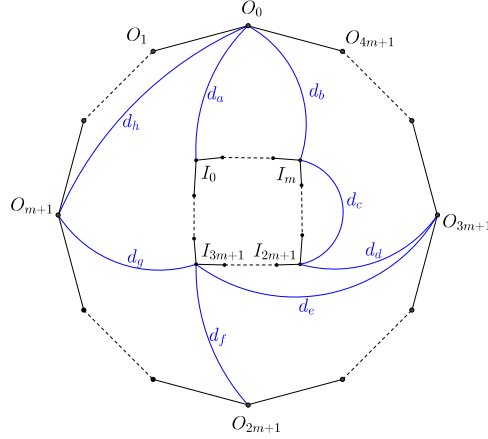
Proof. Observe that (Q, I) is gentle for proposition 7.1, (a), (b) and (c) follow from remark 7.7; (d) is the second condition for being an algebra with root and finally (e) is the statement of proposition 7.15. The converse statement may be proved in much the same way as lemma 7.11. See the following example to illustrate the procedure. \square

7.17. Example. Let (Q, I) be the following bound quiver.



We want to find an $(m + 2)$ -angulation of a certain polygon $P_{p,q,m}$. We start with the diagonal corresponding to the vertex a . Since the arrow $a \rightarrow b$ is clockwise oriented and in a there is not involved any relation, a is in correspondence with a diagonal of type 1, say the diagonal d_a between the vertices O_0 and I_0 and b is in correspondence with another diagonal of type 1, label d_b between the vertices O_0 and I_m . In c there is a counterclockwise internal relation, then c is in correspondence with the diagonal of type 3 between the vertices I_m and I_{2m+1} and d is in correspondence with the diagonal d_d of type 1 between the vertices O_{3m+1} and I_{2m+1} . Since in e there is not a relation and the arrow $d \rightarrow e$ is clockwise oriented, e is in correspondence with a diagonal d_e of type 1 between the vertices O_{2m+1} and I_{3m+1} . Analogously, since in f there is no a relation and the arrow $g \rightarrow f$ is counterclockwise oriented, g is in correspondence with a diagonal d_g of type 1 between the vertices O_{m+1} and I_{3m+1} . Finally, since in h there is a clockwise oriented relation, h is in correspondence with a diagonal of type 2 between the vertices O_{m+1} and O_0 . Therefore we get the

following $(m + 2)$ -angulation of $P_{4,4,m}$ which proves that (Q, I) is the bound quiver of a connected m -cluster tilted algebra of type $\tilde{A}_{4,4}$.



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